Estimation of Computational Error by Higher Order Approximation in the Multipoint Meshless FDM

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Konferencja Użytkowników Komputerów Dużej Mocy
Zakopane, 11 – 13 marca 2015 r.

Introduction

- Error analysis – essential part of b.v. problems solution
- On Multipoint meshless FDM
  - solution approach
- Bases of meshless finite difference method (MFDM) error analysis
- Application of Multipoint method to reference solution generation
- Numerical analysis of benchmark problems
- Final remarks

Objective of this research

Investigation of the **Multipoint** MFDM application to
a posteriori **error analysis** by means of generation
of **high** quality HO **reference** solutions

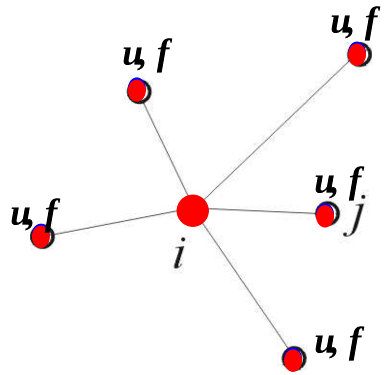
Raising order of local MFDM approximation

ON ERROR ANALYSIS

- Solutions
  - true u^T
  - rough (low order, coarse mesh) u^L
  - higher quality (higher order, fine mesh) u^H
- Generation of reference solution u^H
  - mesh density increase (*h*)
  - raising approximation order (*p*)
  - mixture of both (*hp*)

- Raising order of local MFDM approximation
  - Defect (deferred) correction – use of MFD stars with increased number of nodes
  - Use of (additional) generalised d.o.f.
  - ***Multipoint*** approach
  - Use of higher order (HO) correction terms
  - *p*- and *p/h*- adaptive approach

Idea of Multipoint approach



- Given PDE (ODE)

$$\mathcal{L}u = f, \qquad u = u(P)$$

- FD discretization
  - Standard

$$\mathcal{L}u_i \approx Lu_i = \sum_{j(i)} C_j u_j = f_i \quad \Rightarrow \quad \boxed{Lu_i = f_i} \qquad u_j = u(P_j)$$

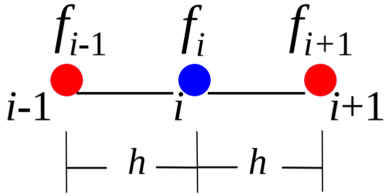
  - Multipoint

$$\mathcal{L}u_i \approx \sum_{j(i)} C_j u_j = \sum_{j(i)} \alpha_j f_j \quad \Rightarrow \quad \boxed{Lu_i = Mf_i}$$

$\mathcal{L}$ – differential operator. In general it may be referred to: differential eqs, boundary conditions, integrand in global formulation of the considered b.v. problem

f_i – value of the whole operator $\mathcal{L}u_i$ or its part only, e.g. a specific derivative $u^{(k)}$

Multipoint – simple examples



Classic FD

$$Au_{i-1} + Bu_i + Cu_{i+1} = f_i$$

Multipoint approach

$$Au_{i-1} + Bu_i + Cu_{i+1} = \alpha f_{i-1} + \beta f_i + \gamma f_{i+1}$$

1st derivative

$$u_i' \approx \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2)$$

$$\frac{u_{i+1} - u_{i-1}}{2h} \approx \frac{(u_{i-1}' + 4u_i' + u_{i+1}')}{6} + O(h^4)$$

2nd derivative

$$u_i'' \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2)$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \approx \frac{(u_{i-1}'' + 10u_i'' + u_{i+1}'')}{12} + O(h^4)$$

Higher order approximation without raising the number of nodes!

Generalization of the classic Multipoint FDM

	Collatz	**Meshless**
• **Mesh**	*regular*	*no mesh, irregular* or *regular* cloud of nodes
• **Formulation**	*local*	*local, global, mixed*
• **Approximation**	*interpolation*	*MWLS approximation*

Method variants

specific

$$\sum_{j(i)} C_j u_j = \sum_{j(i)} \alpha_j f_j$$

general (variants)

$$\sum_{j(i)} C_j u_j = \sum_{j(i)} \alpha_j u_j^{(k)}$$

Multipoint Meshless FDM approach

new Multipoint Meshless FDM =

= Meshless FDM + Formulations +

+ MWLS approximation +

+ Collatz multipoint approach

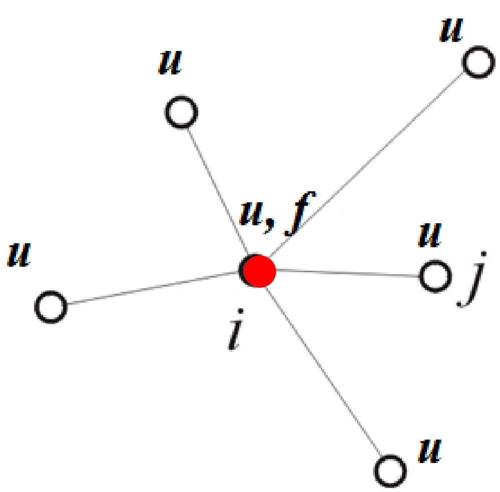
⬇

provides higher order approximation used for

- Solution of b.v. problems
- **A posteriori error analysis**

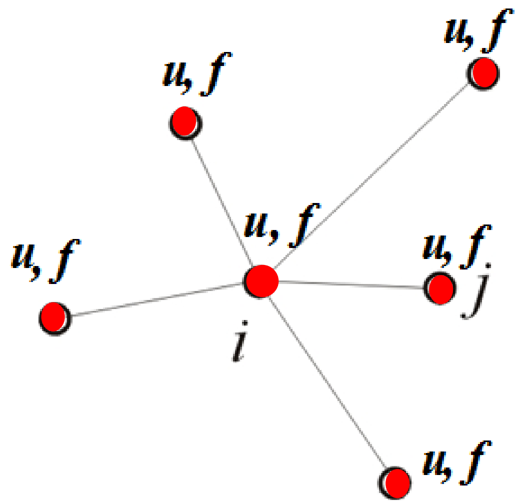
Basic Multipoint formulations

MFD star and d.o.f.



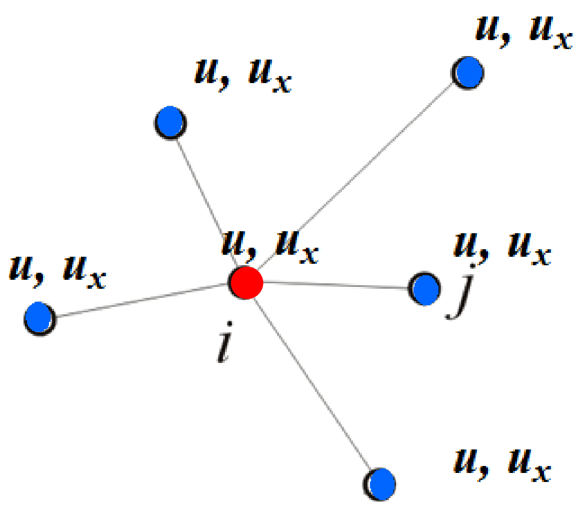
$$\sum_{j(i)} C_j u_j = f_i$$

a) Standard meshless FDM



$$\sum_{j(i)} C_j u_j = \sum_{j(i)} \alpha_j f_j$$

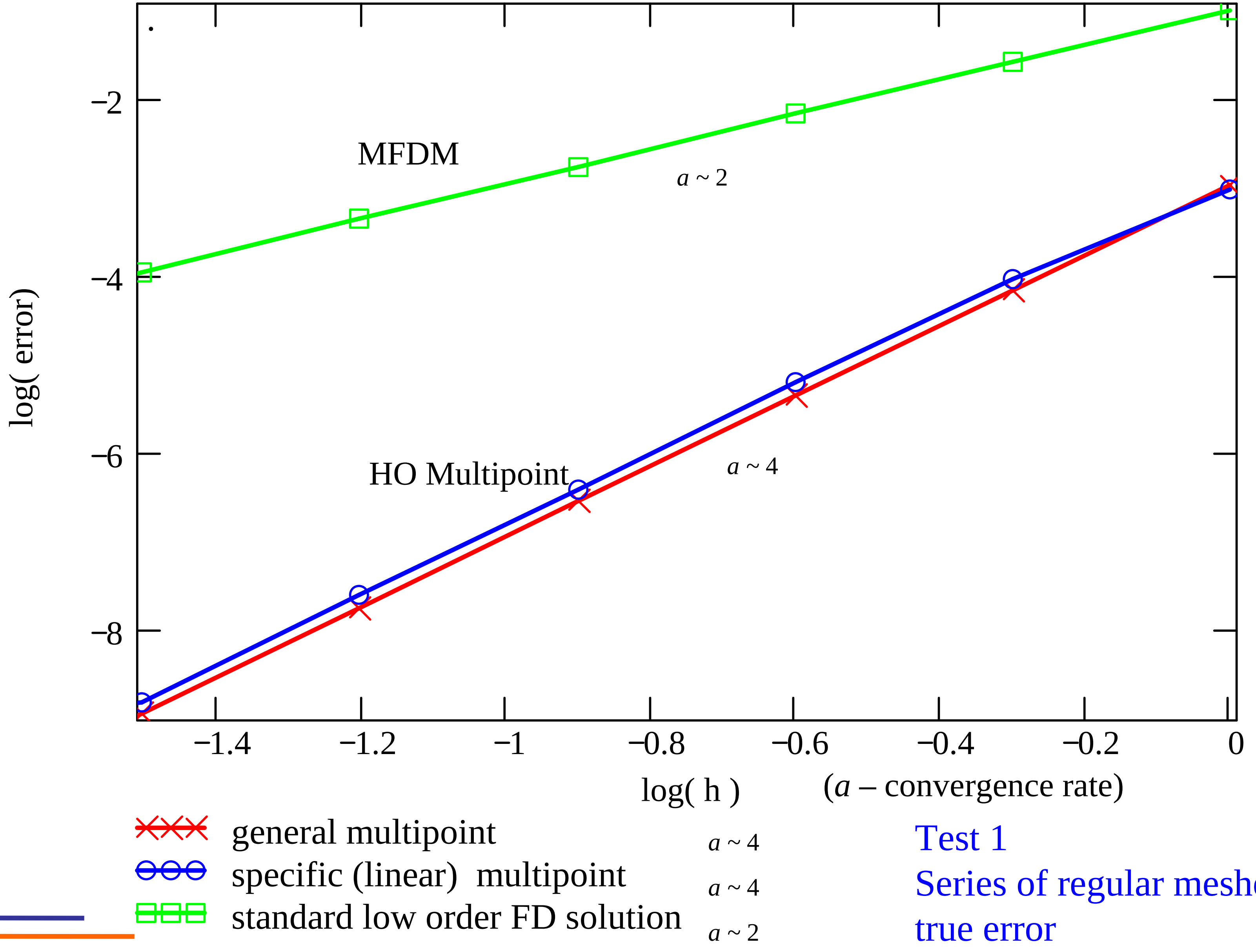
b) Specific multipoint



$$\sum_{j(i)} C_j u_j = \sum_{j(i)} \alpha_j u_j^{(k)}$$

c) General multipoint

Comparison of solution convergence



Test 1

Series of regular meshes

true error

Error analysis – bases

- **Solution types**
  - u^T – true solution (unknown)
  - u^L – rough solution (known)
  - u^H – higher order improved solution usually assumed as the reference one

Error

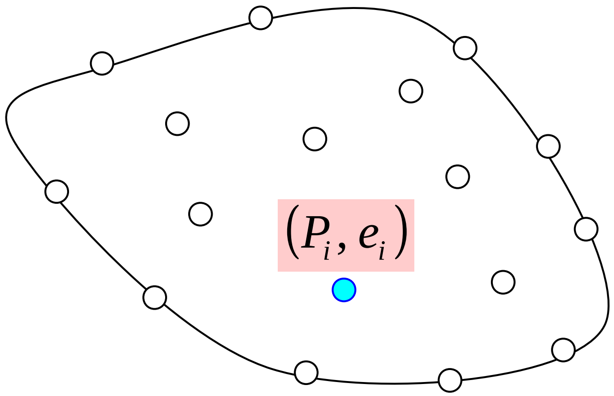
- **Types**
  - a priori, a posteriori (to examine solution quality, to generate adaptive mesh)
  - solution error, residual error
  - local, global
- **Norms**
  - maximum, mean square, energy
- **Definitions**
  - true low order error $e^{TL} = \left\| u^T - u^L \right\|$
  - true higher order error $e^{TH} = \left\| u^T - u^H \right\|$
  - estimated error $e^{HL} = \left\| u^H - u^L \right\| \approx e^{TL}$

- **Problem:** how to generate a reference solution u^H? **Multipoint MFDM**

A posteriori error MFDM analysis – local error estimation

True solution

$$\mathcal{L}u = f, \qquad \longrightarrow \quad u^T$$



Lower order solution (standard MFDM)

$$\mathcal{L}u \approx Lu = f, \qquad Lu = f \qquad \longrightarrow \quad u^L$$

Higher order solution (Multipoint MFDM)

$$\mathcal{L}u \approx Lu = Mf, \qquad Lu = Mf \qquad \longrightarrow \quad u^H$$

Local solution error

$$e^{TL} = u^T - u^L$$
$$e^{TH} = u^T - u^H$$

$$\longrightarrow \quad \boxed{e^{HL} = u^H - u^L \approx e^{TL}}$$

Higher order (**Multipoint**) estimation of the local solution error

$$\boxed{e^{HH} = u^{H(p1)} - u^{H(p2)} \approx e^{TH}}$$

various use of HO approximations

Local residual error

$$r = \mathcal{L}u - f \quad \longrightarrow$$

$$r^L = Lu^L - f$$

Standard – low order estimation of the local residuum

$$\boxed{r^H = Lu^H - f}$$

Improved – higher order estimation of the local residuum

A posteriori error MFDM analysis – global error estimation

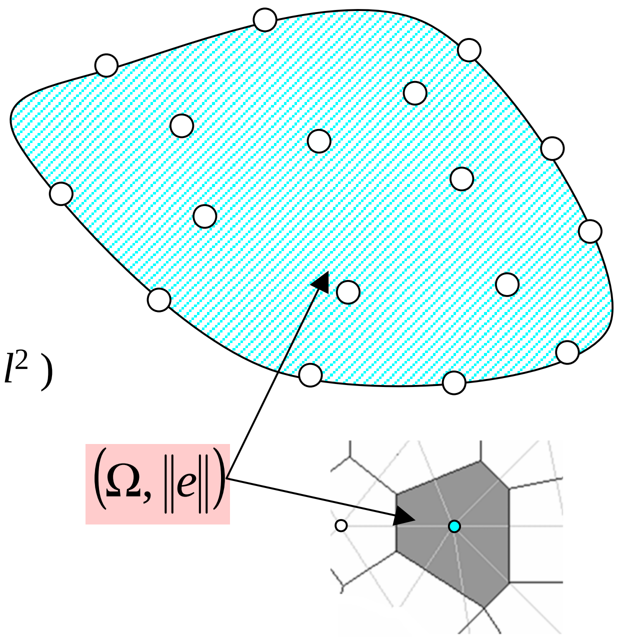
Global solution error $\eta = \|e\|$

- Error norms ||·|| used:

$$\|e\|_E = \sqrt{\frac{1}{\Omega}\int_\Omega b(e,e)\,d\Omega}$$ energy norm

$$\|e\|_2 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(e_i)^2}$$ mean square norm (discrete l^2)

Effectivity index of estimator $I_{eff} = 1 + \left|\frac{\eta - \|e^{TL}\|}{\|e^{TL}\|}\right|$



Hierarchic estimators

H-type $e = u_h^L - u_{h/2}^L$

P-type $e = u_p^L - u_{p+1}^L$

MHO-type $e = u_p^L - u_{p+s}^H$

Smoothing estimators

ZZ-type $e = u'^L_{rough} - u'^L_{smooth}$

MHO-type $e = u'^L_p - u'^H_{p+s}$

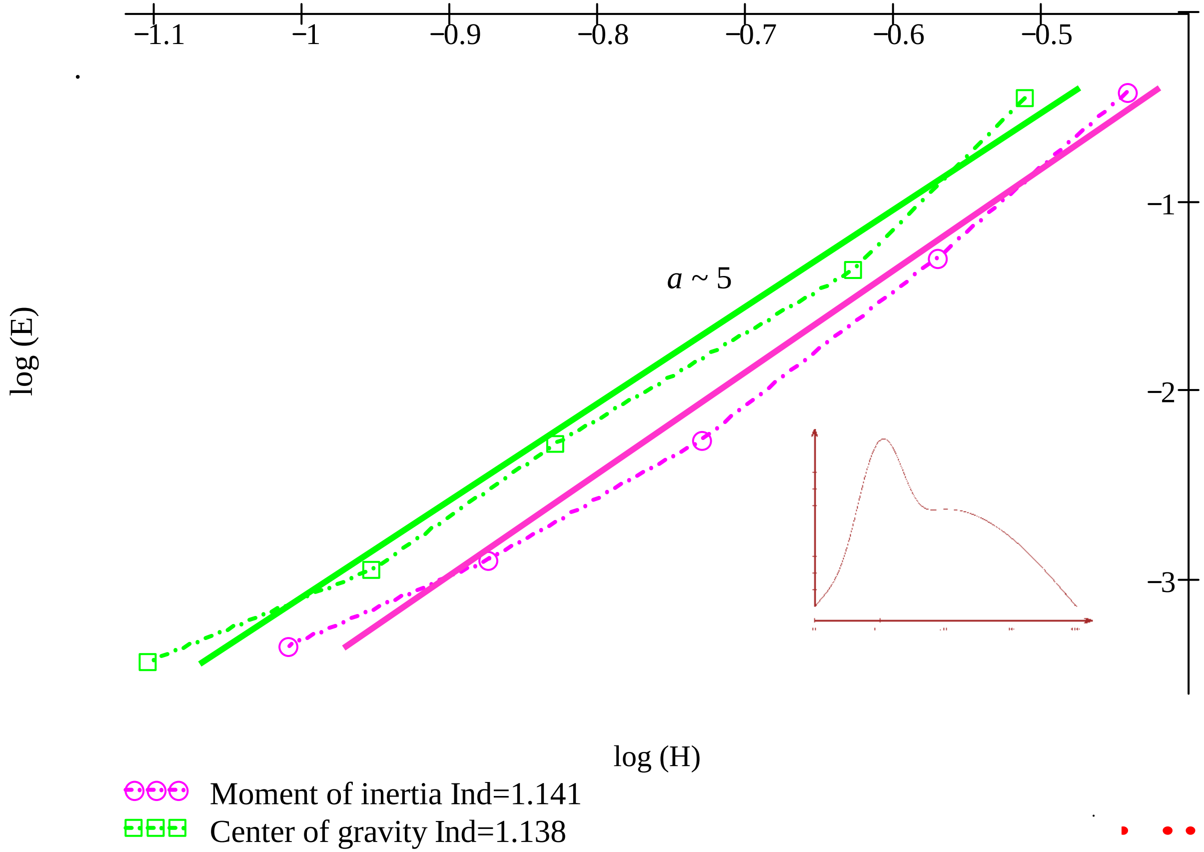
Residual estimators

Explicit $e = \sqrt{h^2\|r\|^2 + h/2\|J\|^2}$

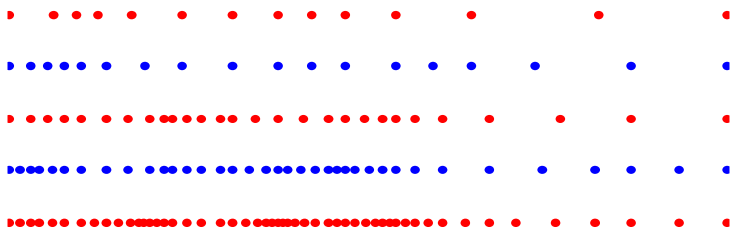
Implicit $e \quad \rightarrow \quad b(e,e) = r$

Error analysis

- **Error indicators for irregular meshes**



Series of adaptive meshes



Test 2. Solution convergence based on the series of adaptive meshes using the both types of the error indicators

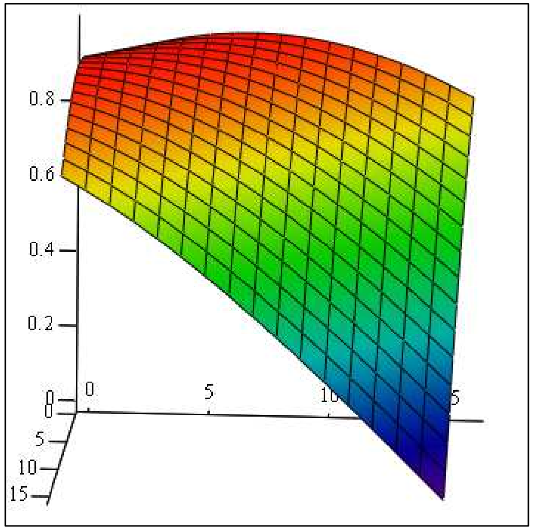
Benchmark 2D problems

Two-dimensional Poisson's b.v. problem

$$\nabla^2 u = f(x, y) \quad \text{w} \quad \Omega$$

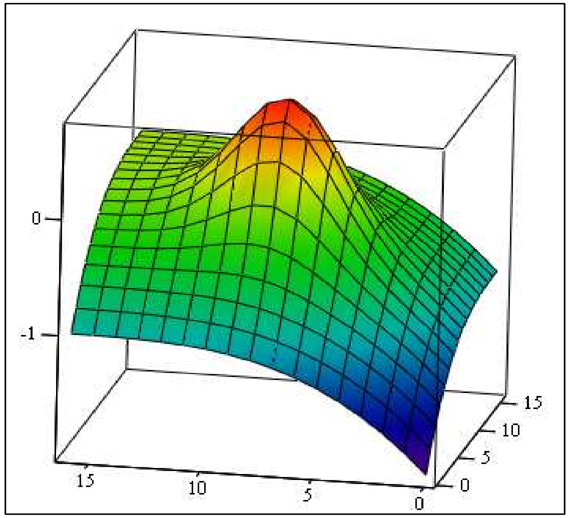
with the Dirichlet b.c. $0 \le x \le 1, \quad 0 \le y \le 1$

Test 3



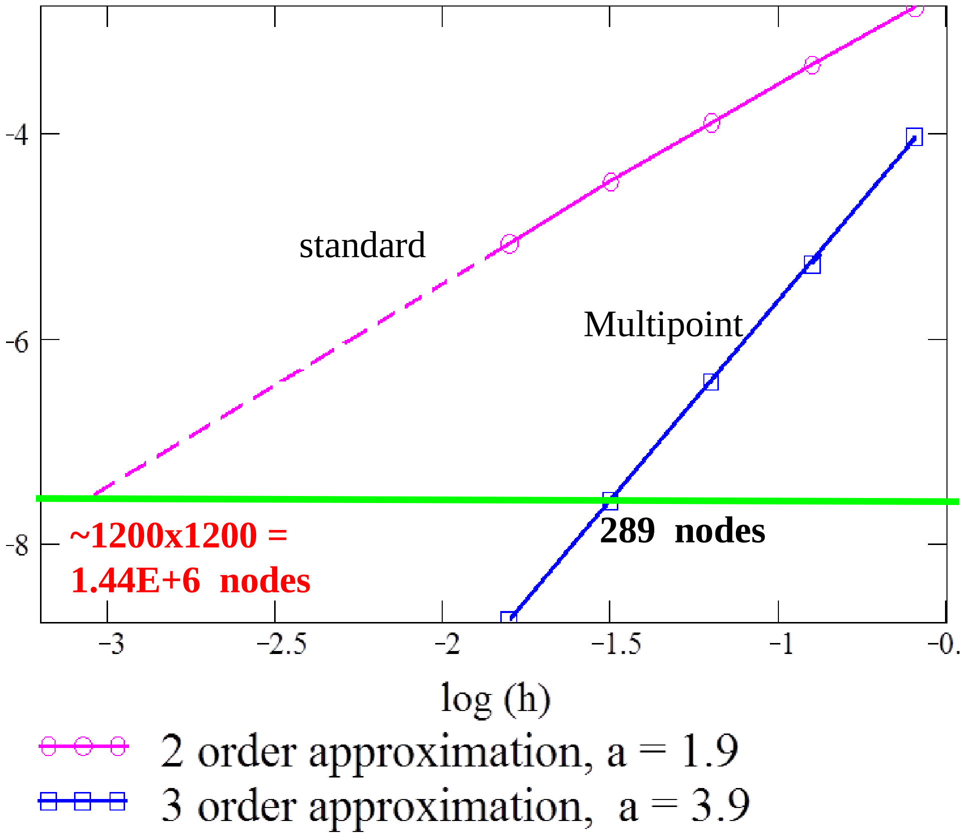
$$u(x,y) = \sin(x + y)$$

Test 4

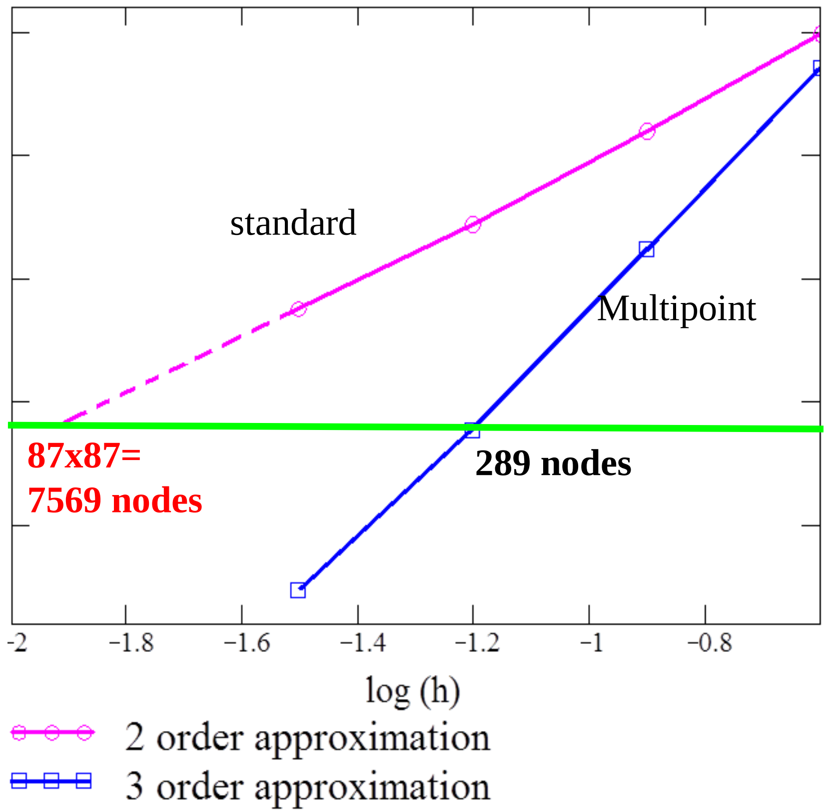


$$u(x,y) = -x^3 - y^3 + \exp\left(-100(x-0.5)^2 - 100(y-0.5)^2\right)$$

Convergence for multipoint & standard MFDM



Test 3

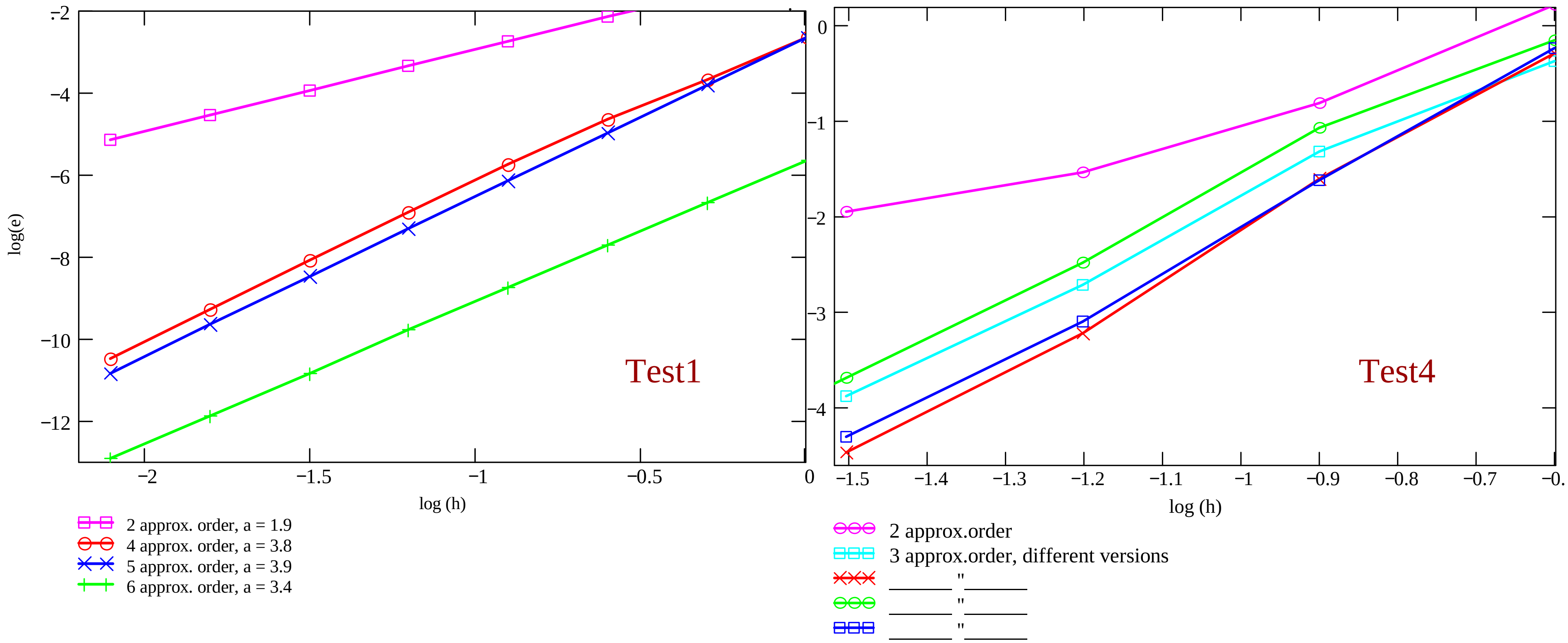


Test 4

Solution convergence, results of the general multipoint method and the standard MFDM

Multipoint error analysis

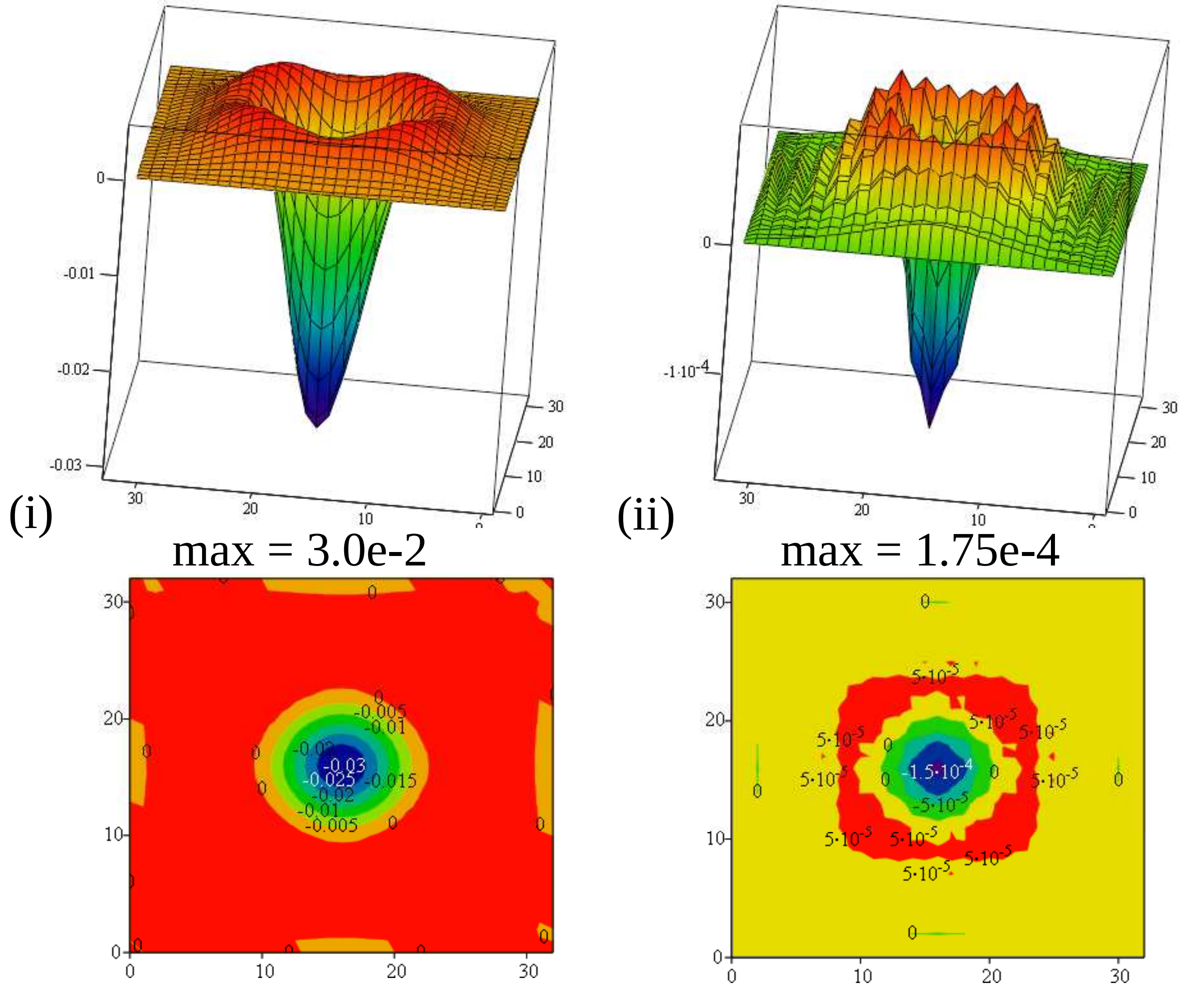
- **Hierarchic (*p*-type)**



Solution convergence based on of the exact mean solution errors for 1D and 2D tests and various approximation orders

Multipoint error analysis

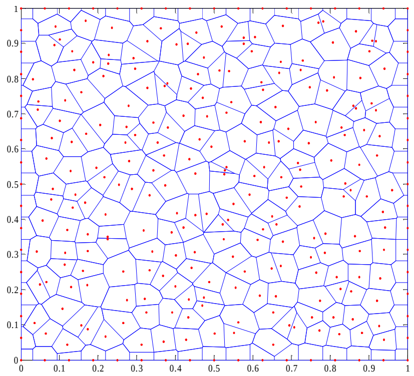
- **Hierarchic (*p*-type)**

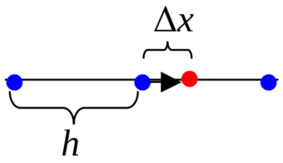


Test 4. The exact solution error of (i) basic MFDM; (ii) Multipoint MFDM

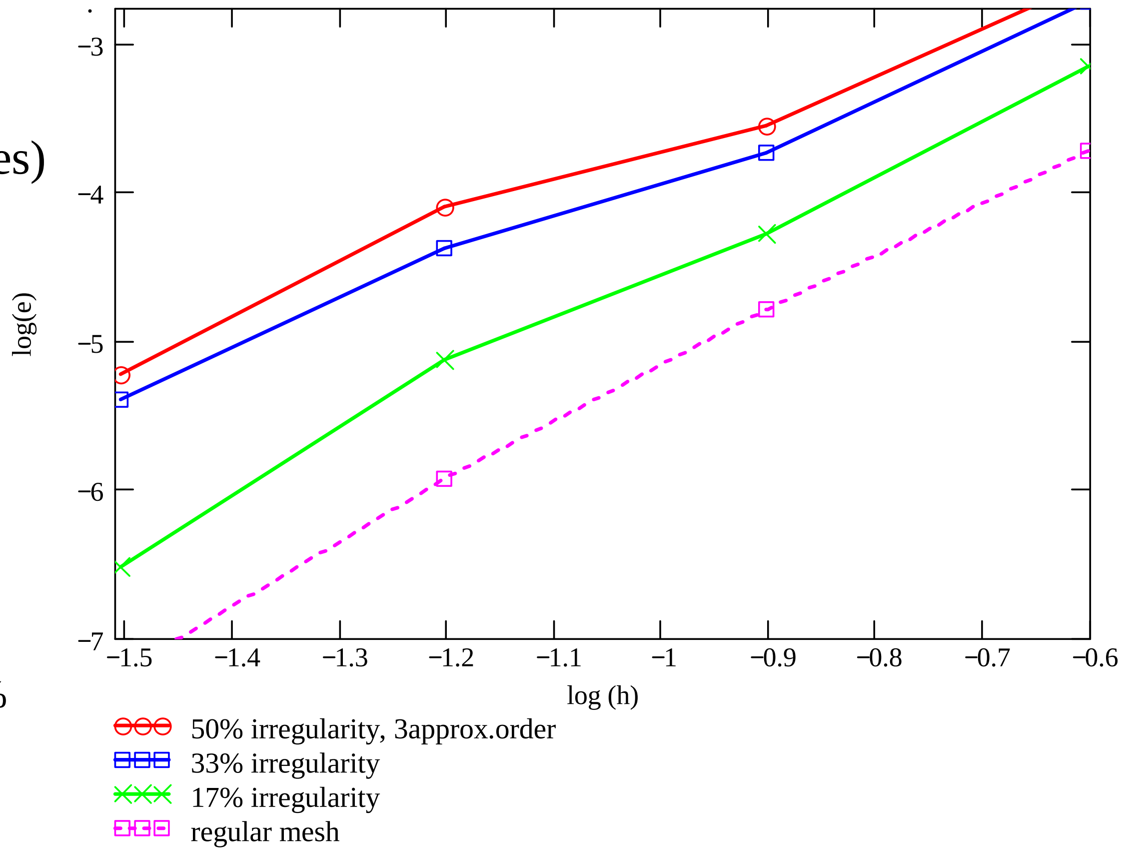
Multipoint error analysis – irregular meshes

Voronoi polygons for
random irregular mesh
(up to 33% irregularity 289 nodes)





$$\frac{\Delta x}{h} \leq \frac{1}{3} = 33\%$$

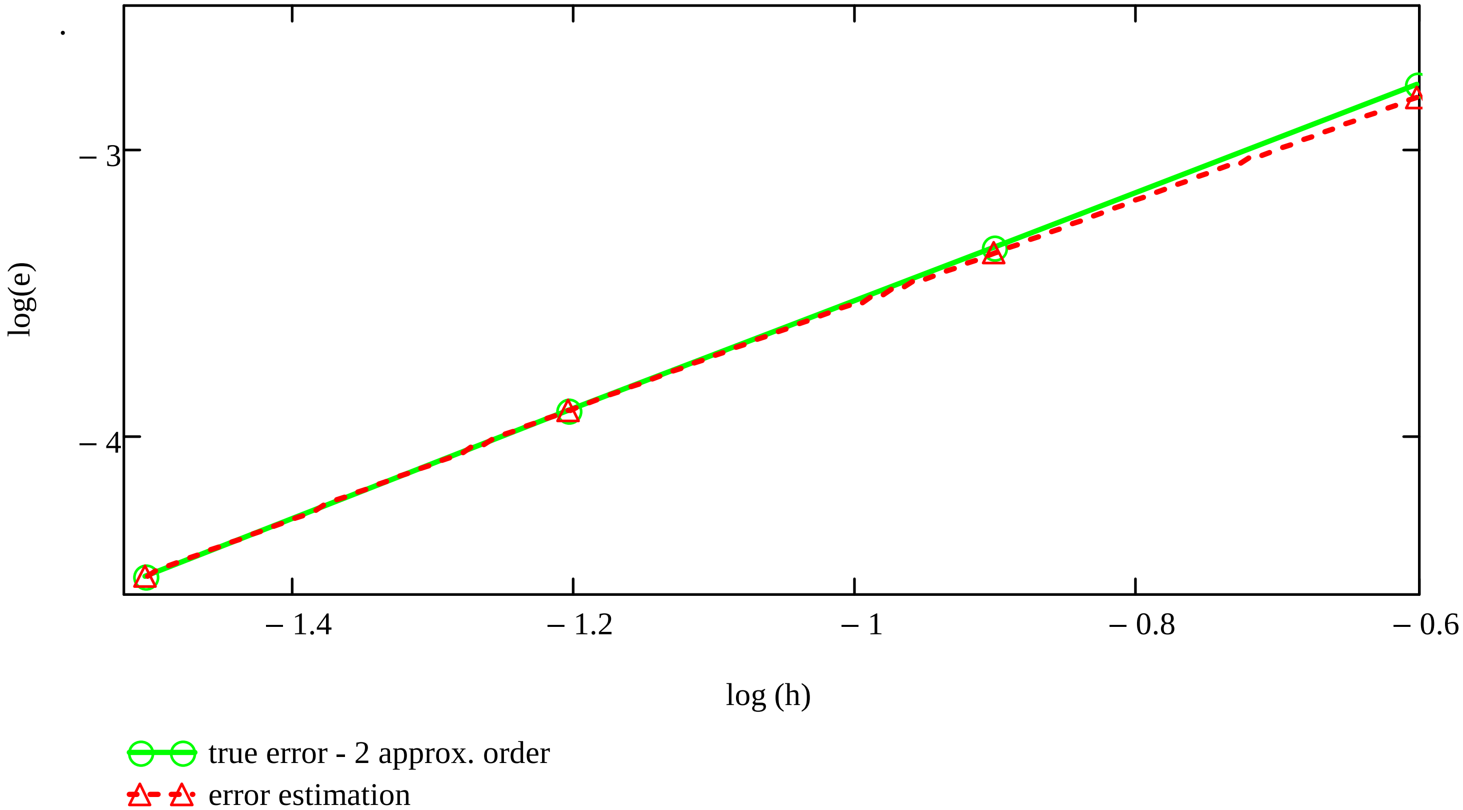


Test 3. Convergence of the average of 20 random irregular meshes and regular ones, all using 3rd approximation order

Multipoint error analysis

- **Hierarchic estimator**

$$I_{eff} = 1 + \left| \frac{\left\| e^{HL} \right\| - \left\| e^{TL} \right\|}{\left\| e^{TL} \right\|} \right|$$



Comparison of the error estimation e^{HL} and the exact error e^{TL} for low order solution. Test 3, general Multipoint approach. Effectivity index ~1.1

Multipoint error analysis

- **Effectivity index**

$$I_{eff} = 1 + \left| \frac{\|e^{HL}\| - \|e^{TL}\|}{\|e^{TL}\|} \right|$$

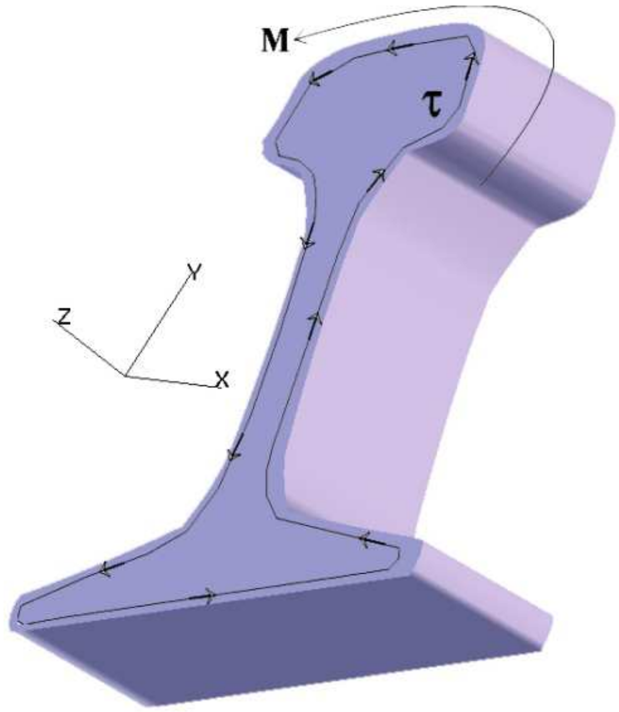
True solution error	Test 3		Test 4	
	Max norm	**Euclidean norm**	**Max norm**	**Euclidean norm**
Higher order true error e^{TH} (version 2)	3.0e-6	1.18e-6	5.12e-3	1.94e-3
Higher order true error e^{TH} (version 3), local formulation	1.22e-6	4.12e-7	1.48e-3	5.93e-4
Higher order true error e^{TH} (version 3),global formulation	1.8e-6	8.29e-7	8.05e-3	3.11e-3
Higher order true error e^{TH} (version 4)	4.0e-6	2.08e-6	6.33e-3	3.5e-3
Lower order true error e^{TL}	2.54e-4	1.24e-4	1.35e-1	2.18e-2
Error estimation e^{HL}	2.5e-4 – 2.54e-4	1.22e-4 – 1.24e-4	1.27e-1 – 1.34e-1	1.83e-2 – 2.12e-2
Effectivity index I_{eff}	**1 – – 1.02**	**1 – – 1.02**	**1.01– – 1.06**	**1.03 – – 1.16**

Error estimation for various Multipoint MFDM versions. Regular 249 nodes mesh

Multipoint error analysis –
prismatic bar twisting

Φ – Prandtl stress function, $G\theta = 1$ – torsional stiffness, Ω – domain of the bar cross-section.

$$\begin{cases} \nabla^2\Phi = -2G\theta, & \text{in} \quad \Omega \\ \Phi = 0, & \text{on} \quad \partial\Omega \end{cases}$$



The total shear stress

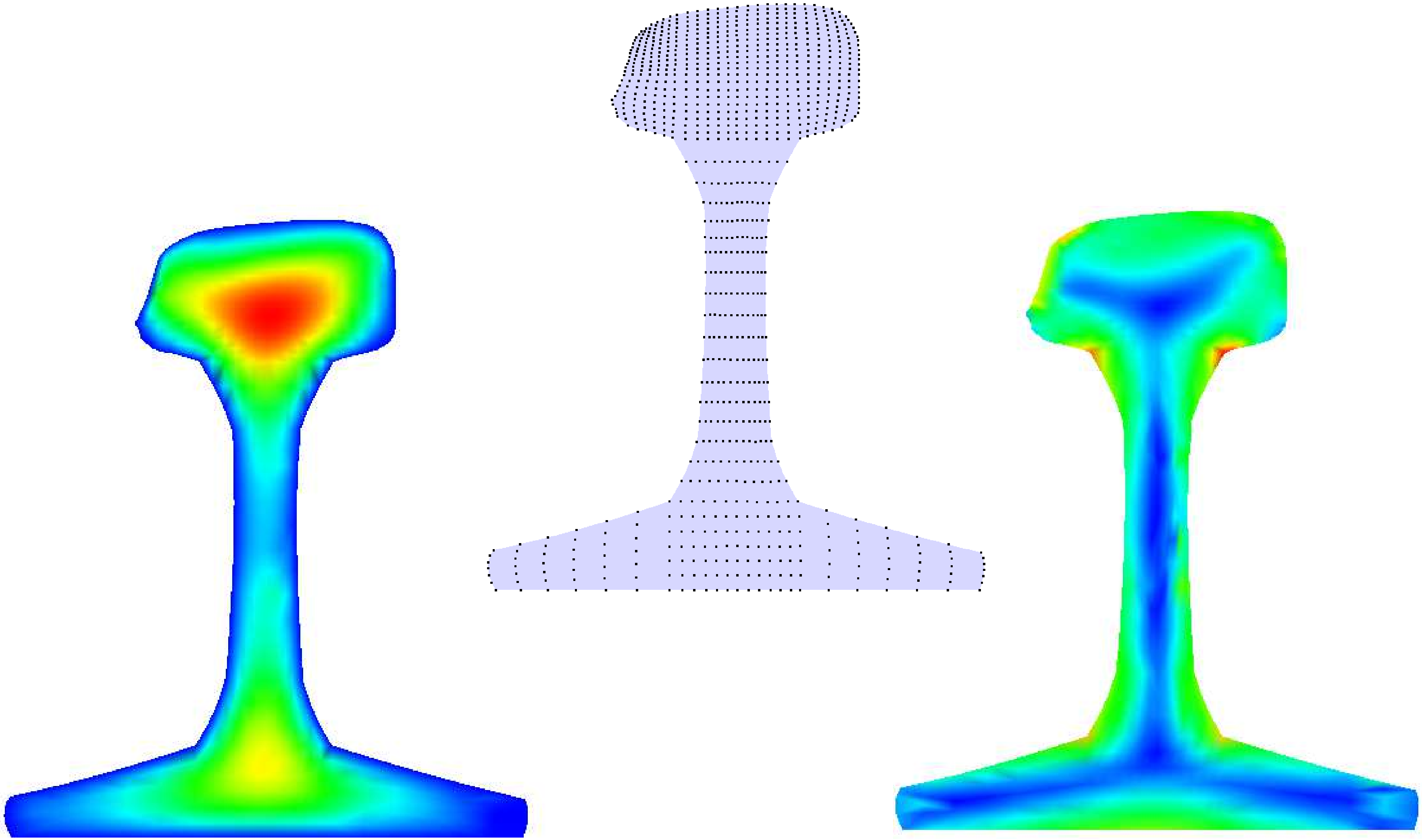
$$\tau = |\, grad\ \Phi \,| = \sqrt{\tau_{zx}^2 + \tau_{zy}^2}$$

shear stresses are

$$\tau_{zx} = \frac{\partial\Phi}{\partial y}, \qquad \tau_{zy} = -\frac{\partial\Phi}{\partial x}$$

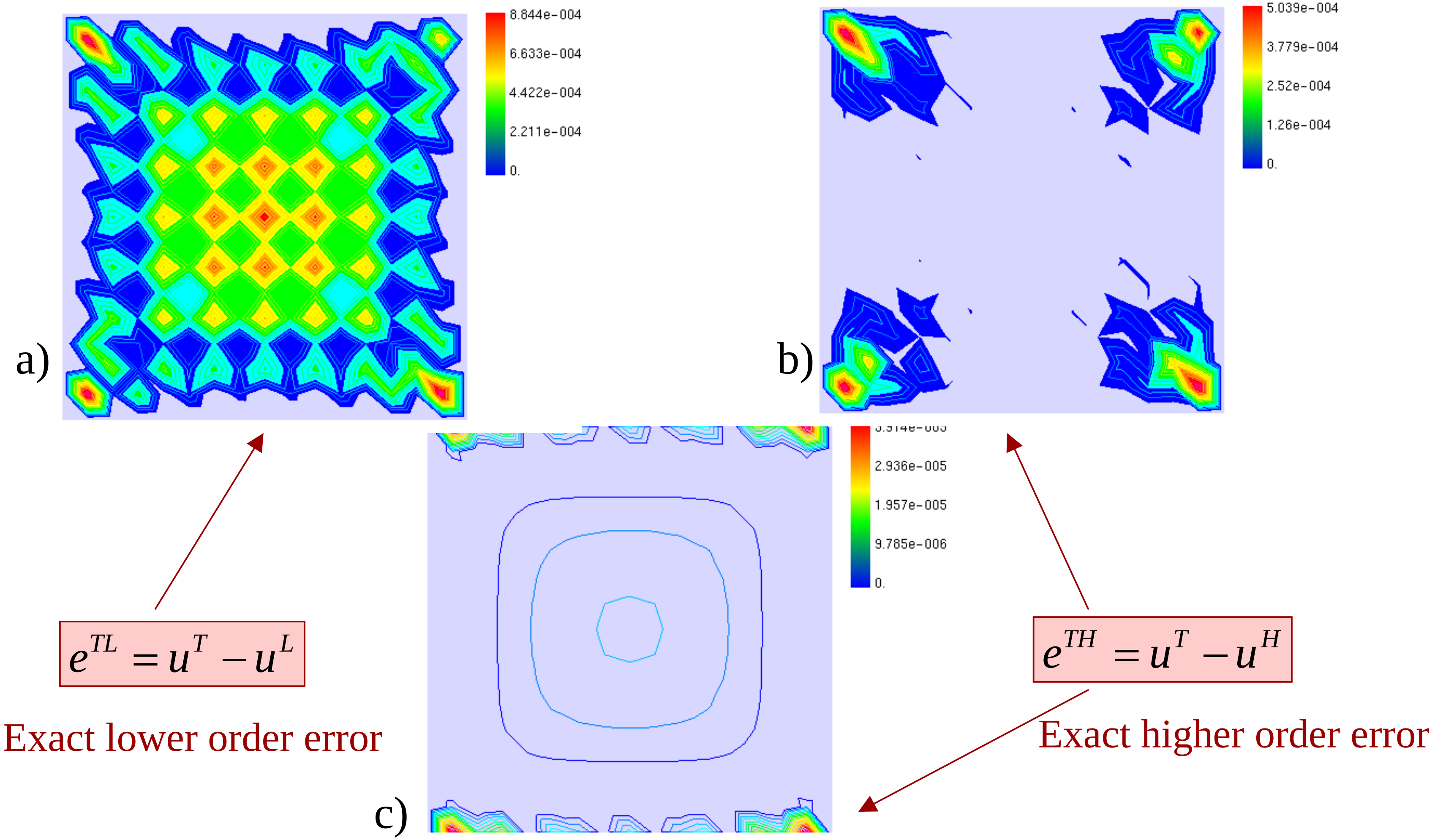
- **Railroad rail cross-section**
- **Benchmark test – square cross-section**

Prismatic bar (rail-shape) twisting



Irregular mesh, Prandtl stress function and total shear stress in railroad rail shape bar

Prismatic bar twisting.
True solution error for Prandtl function

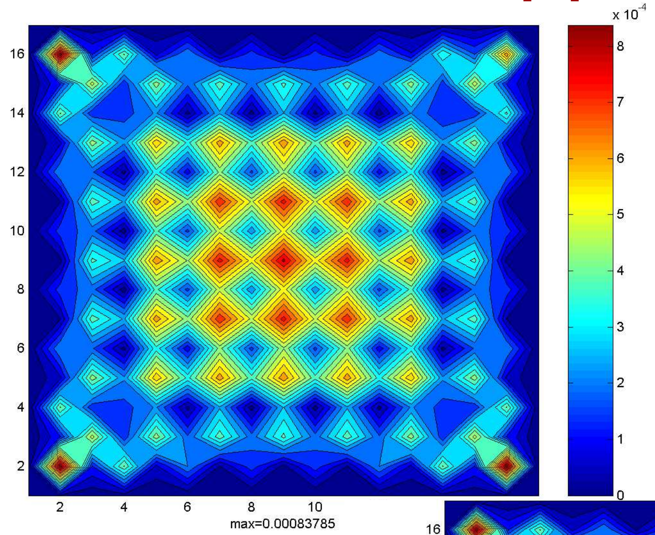


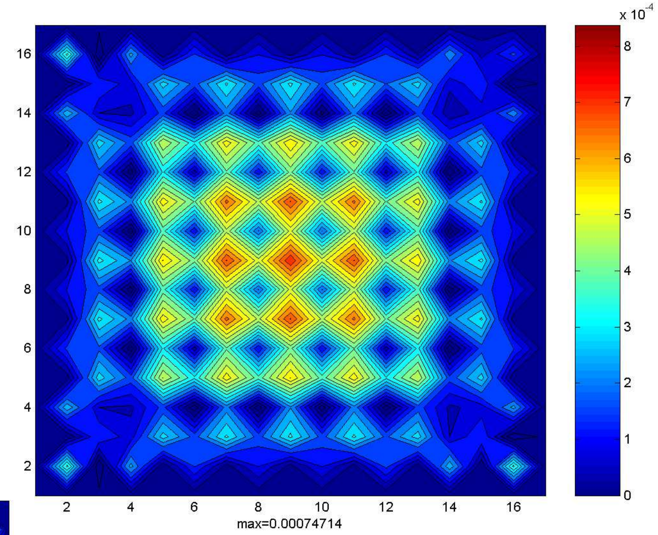
Multipoint solution error:

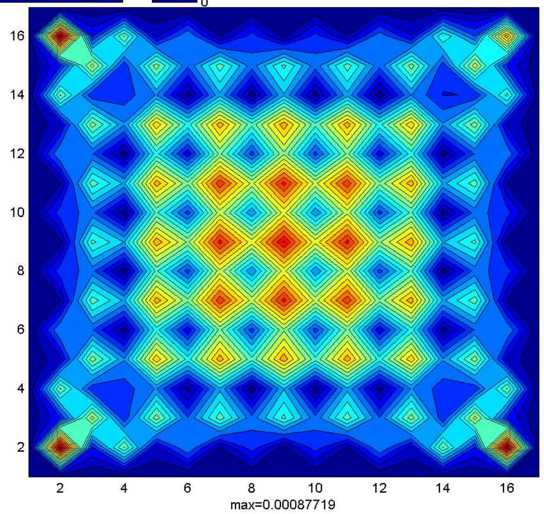
a) local formulation and 2nd app.order, b) local formulation and 3rd app.order,

c) global-local formulation (MLPG5) and 3rd app.order

Estimated 2nd approx. order solution error

a) 

b) 

c) 

$$e^{HL} = u^{H} - u^{L} \approx e^{TL}$$

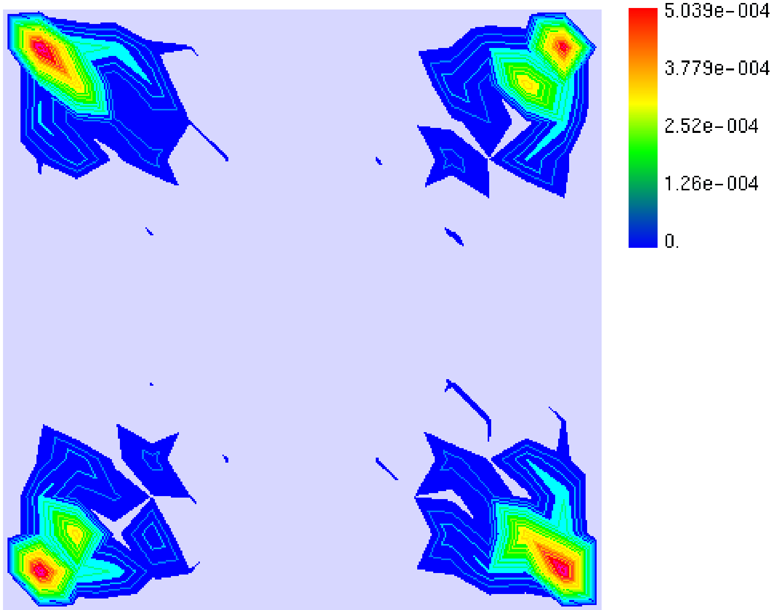
Estimated error

Multipoint solution (local formulation) error:

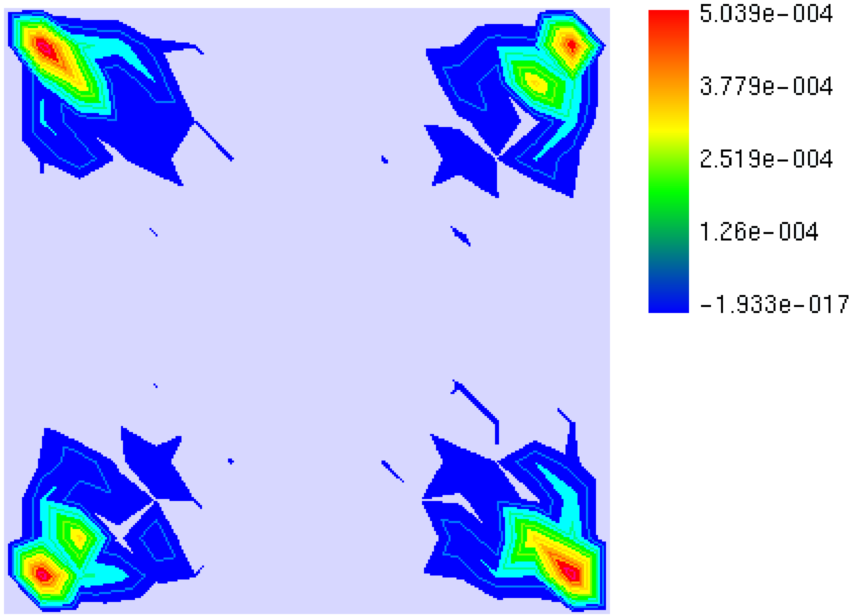
a) exact error of the 2nd app.order b) estimated error by 3rd app.order

c) estimated error by the MLPG5 3rd app.order

Estimated 3rd approx. order solution error



a)



b)

$$e^{HH} = u^{H(p1)} - u^{H(p2)} \approx e^{TH}$$

Estimated error

Multipoint solution error:

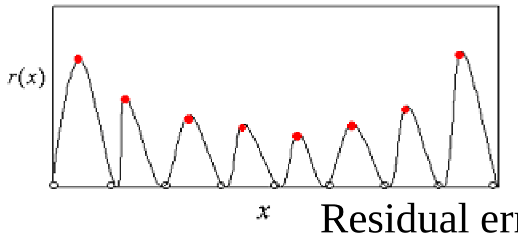
a) exact solution error for the local formulation and 3rd app.order

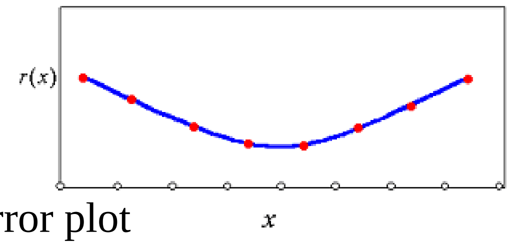
b) solution error for the local formulation and 3rd app.order estimated by the MLPG5

Multipoint error analysis

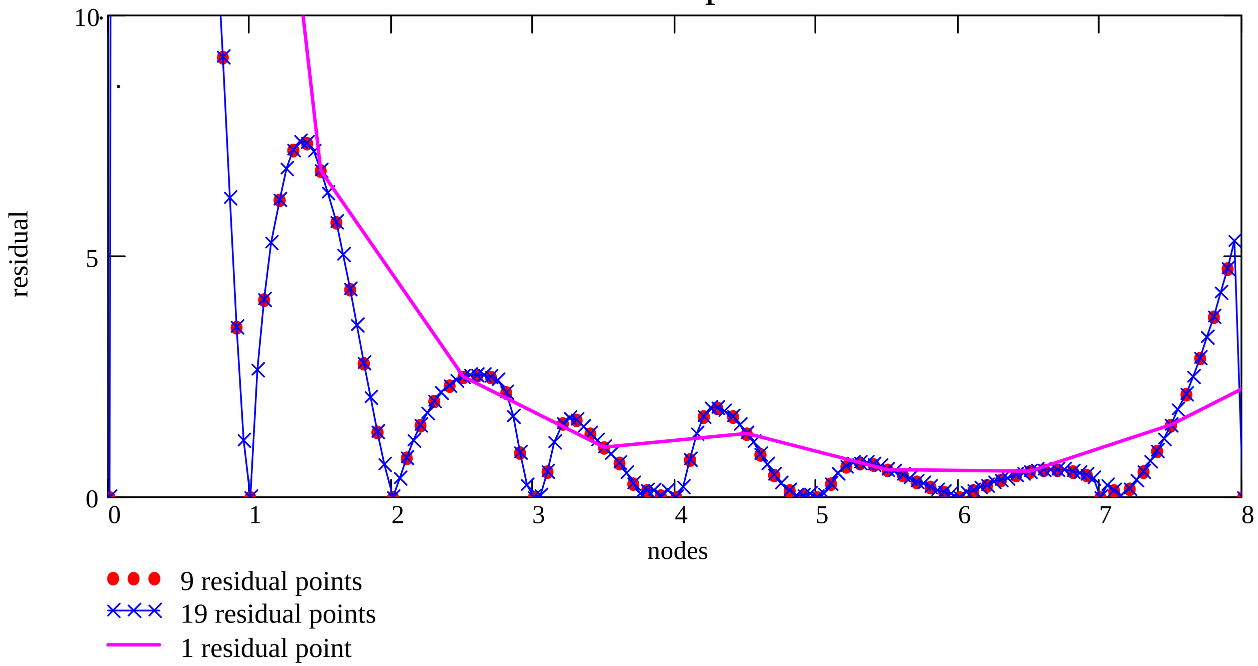
- **Residual error distribution**

$$r^L = Lu^L - f \qquad r^H = Lu^H - f$$





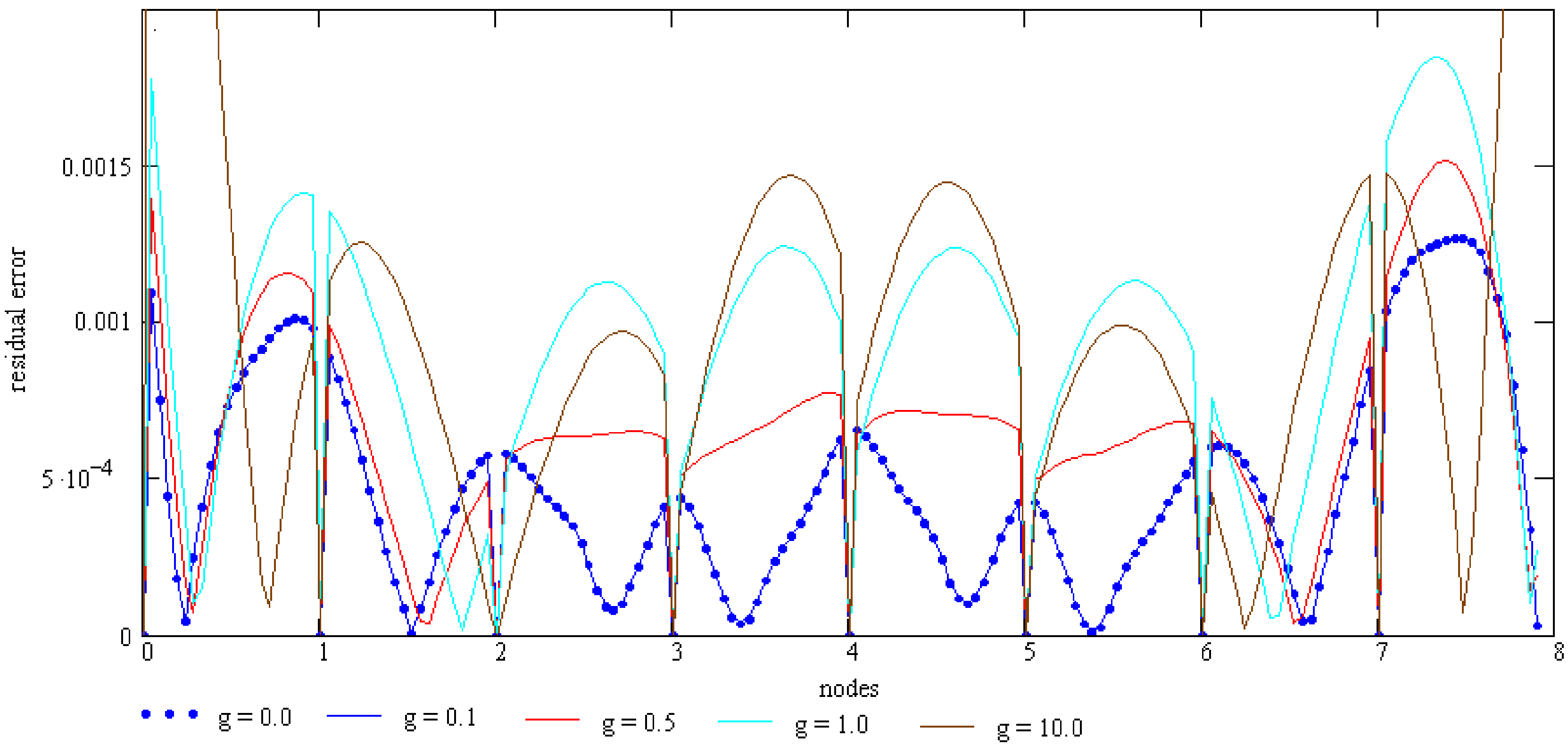
Residual error plot



Residual error distribution using 1, 9, and 19 points between nodes. Test 2.

A posteriori error analysis

- **Residual error**



Residual error distribution using 19 points between nodes.
The influence of the MWLS weighting factor *g*
Smaller *g* provides smaller errors

$$^{k}w_{ij}^{2} = \left(\rho_{ij}^{2} + \frac{g^{4}}{\rho_{ij}^{2} + g^{2}} \right)^{-p+k-1}$$

Multipoint error analysis

Specific case

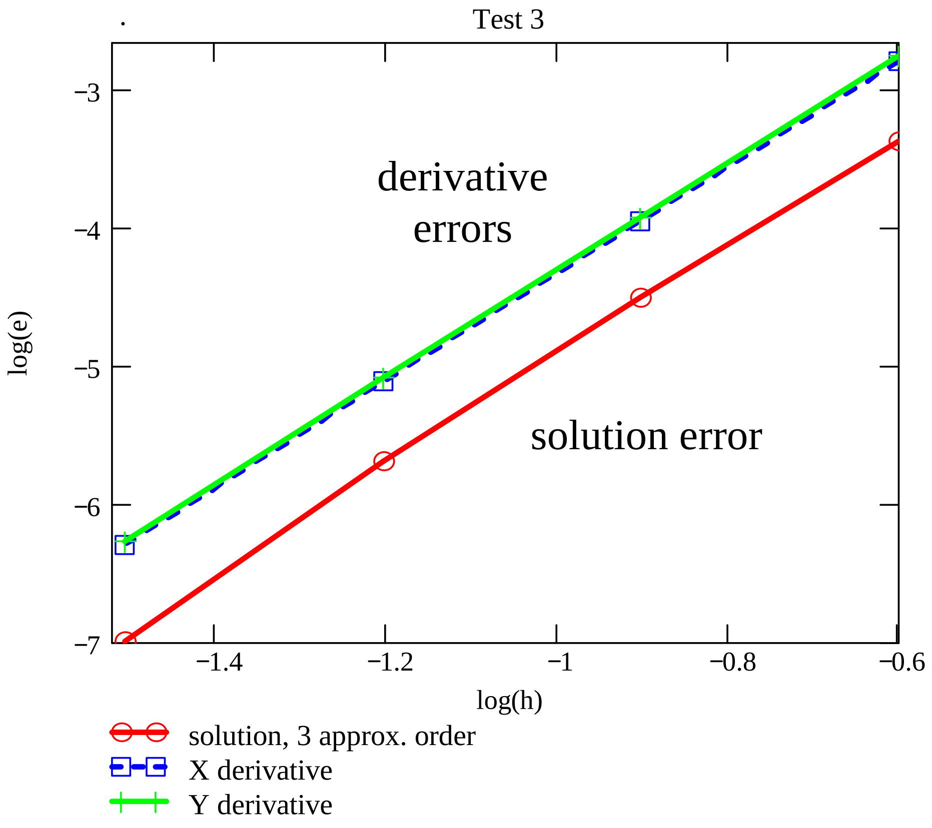
$$\mathbf{u}^{(k)} = \mathbf{C}\,\mathbf{u} + \boldsymbol{\alpha}\,\mathbf{f}$$

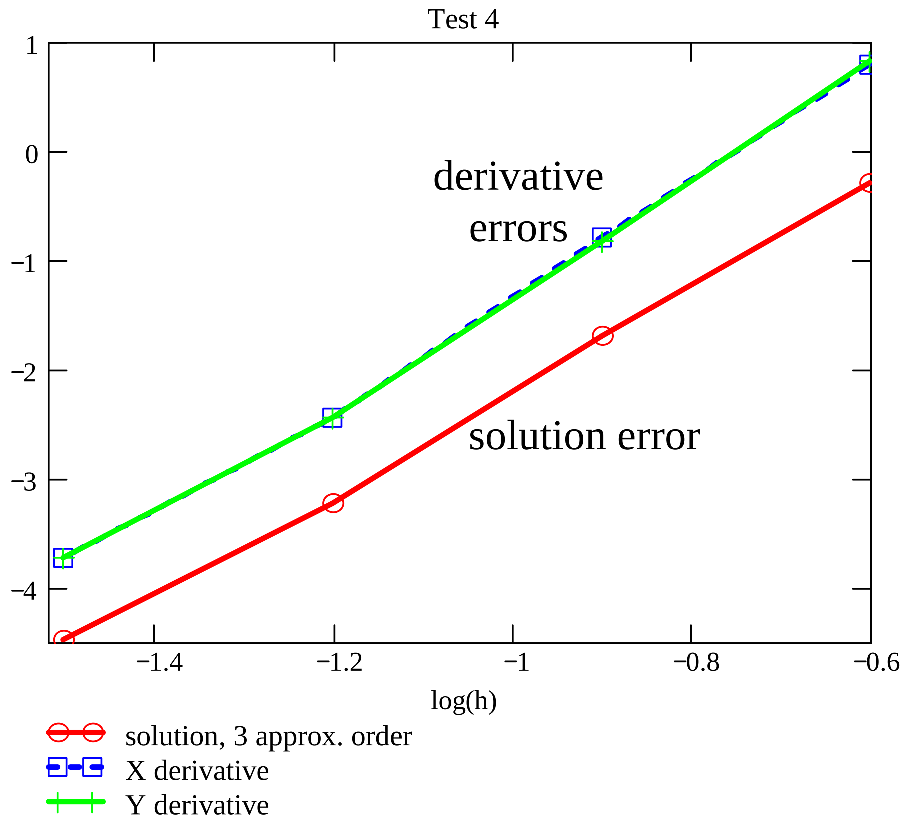
General case

$$\mathbf{u}^{(k)} = \mathbf{A}\cdot\mathbf{u}$$

- **Smoothing**

Zienkiewicz-Zhu type error estimator $e^{TH} = u'^{T} - u'^{H}$ $e^{LH} = u'^{L} - u'^{H} \approx e^{TL}$





Derivatives – true error convergence

Final remarks

- Presented was a posteriori error analysis based on the Multipoint Meshless Finite Difference Method (MMFDM).
- The MMFDM may provide high quality reference solutions for a posteriori error estimation.
- The following solution combinations may be done:
  - HO multipoint one vs. lower order standard (MFDM or FEM) solution $u^H - u^L$
  - Two different HO multipoint solutions, e.g. p+1 and p+2, presenting various use of higher order approximations
- Various types of a posteriori error estimation, including hierarchical, residual and Zienkiewicz-Zhu approaches, may be carried out when supported by the Multipoint MFD method, and applied to both irregular and regular meshes.
- Numerous tests carried out present encouraging, high quality results of such error analysis. However, results of the residual error analysis carried out between nodes show strong dependence of the error value on the choice of the used MWLS smoothing weight factor value *g*.
- Further development and testing of this approach seems to be justified.

Thank you very much for your attention!