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# ON MESH REFINEMENT FOR EVOLUTIONARY ALGORITHMS ACCELERATION APPLIED TO LARGE NON-LINEAR CONSTRAINED OPTIMIZATION PROBLEMS

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## Contents

# Introduction

## Research motivation

A variety of **engineering** and **scientific** tasks may be formulated as **large, non-linear, constraint optimization** problems, e.g.:

- **residual stress** analysis in railroad rails and vehicle wheels,
- **physically** based **approximation** of **experimental data**.

Their effective solution is crucial for various practical **engineering** problems.

**Solution methods** used (convex, and non-convex problems):

**deterministic** like **FDM** (Feasible Directions Method) or Penalty Methods and/or
**probabilistic** like **AI** (e.g. **Evolutionary Algorithms**)

## Research objective

Significant **acceleration** of the **EA** applied to **large, non-linear constraint optimization** problems, where a **function** (given e.g. by its **nodal values**) is searched.
The **speed-up** is based on:

- choice of the **most effective combination** of the evolutionary operators: selection, crossing-over, mutation,
- use of several **new** simple **speed-up techniques proposed** here,
- further **development** of chosen **existing** methods.

# Typical formulations of these optimization problems

## (i) Residual stress analysis

Find minimum of the total **complementary energy**

$$\min_{\sigma^s} W(\sigma^s - \sigma^{R_0}) , \qquad W(\sigma^s - \sigma^{R_0}) = \tfrac{1}{2}\int_V (\sigma^s - \sigma^{R_0})^t C (\sigma^s - \sigma^{R_0})$$

for self-equilibrated stresses $\sigma^s$ satisfying

**equilibrium** equations

$$\partial \sigma^s = 0 \qquad in \quad V$$

**boundary** conditions

$$v^t \sigma^s = 0 \qquad on \quad \partial V$$

and **yield** constraint

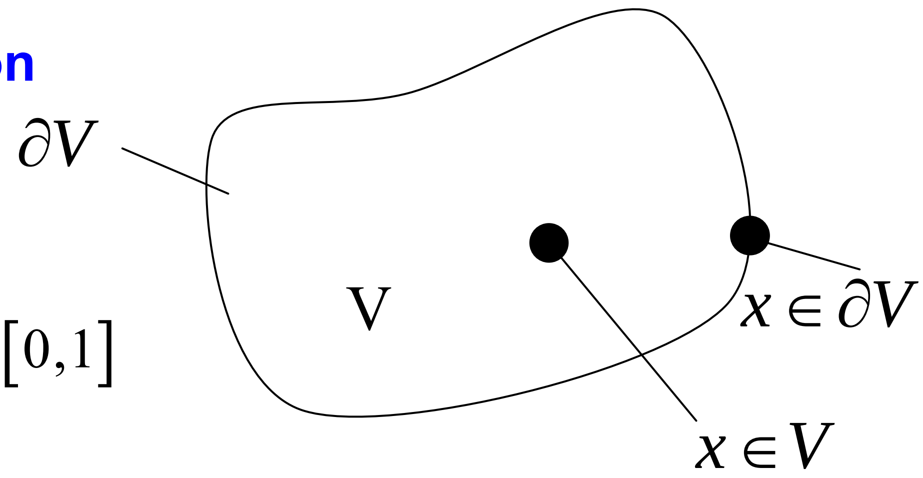
$$\Phi(\sigma^s + \sigma^E) \le 0 \qquad in \quad V \cup \partial V$$

where

- $W$ - total **complementary energy** functional corresponding to stresses $\sigma$ in the whole body $V$
- $\sigma^{R_0}$ - initial self-equilibrated stresses
- $C$ - compliance matrix

## Typical **formulations** of these optimization problems

### (ii) **Physically** based **approximation**




Find $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x)$ that yields the **stationary value** of the **functional**

$$\Phi = (1-\lambda)\,\bar{\Phi}^{T} + \lambda\,\bar{\Phi}^{E} \qquad \lambda \in [0,1]$$

and satisfy

**equality** constraints (theoretical)

$$A(\boldsymbol{\sigma}) = 0$$

**inequality** constraints (experimental)

$$B(\boldsymbol{\sigma}) \le e$$

where

$$\bar{\Phi}^{T} = \frac{\Phi^{T}}{\Phi^{T}_{REFERENCE}}$$ - dimensionless **theoretical** part of the weighted functional $\Phi^{T}$

$$\bar{\Phi}^{E} = \frac{\Phi^{E}}{\Phi^{E}_{REFERENCE}}$$ - dimensionless **experimental** part of the weighted functional $\Phi^{E}$

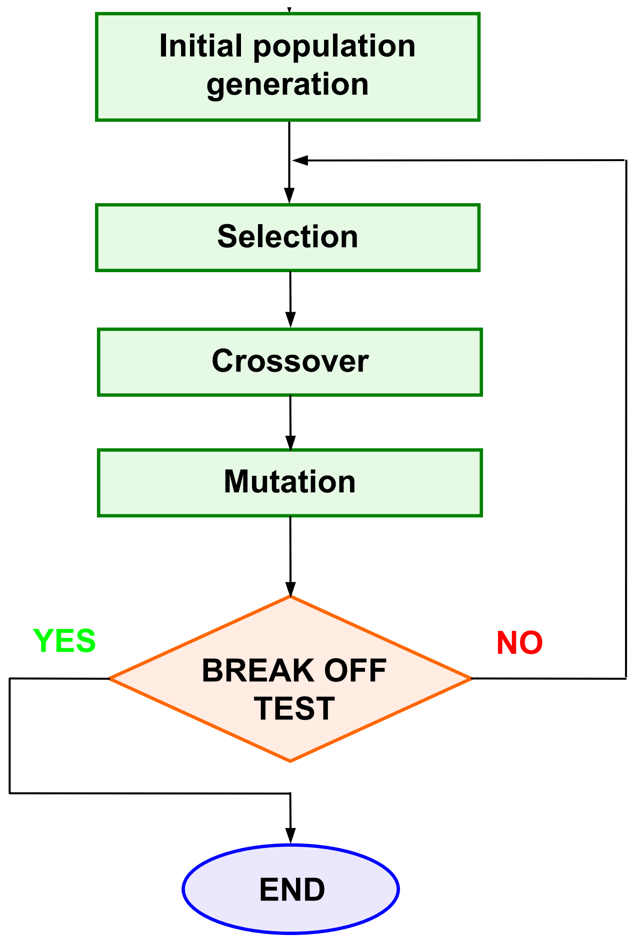
$\lambda \in [0,1]$ - weighting factor

$e$ - admissible tolerance

# Standard Evolutionary Algorithm used

Choice of the **most effective combination** of evolutionary operators

- **Selection** operators:
  - **rank**
  - tournament
- **Crossing-over** operators:
  - simple
  - arithmetic
  - **heuristic**
- **Mutation** operators:
  - uniform
  - **non-uniform**
  - boundary



## Acceleration techniques considered

**Newly** proposed simple acceleration techniques:

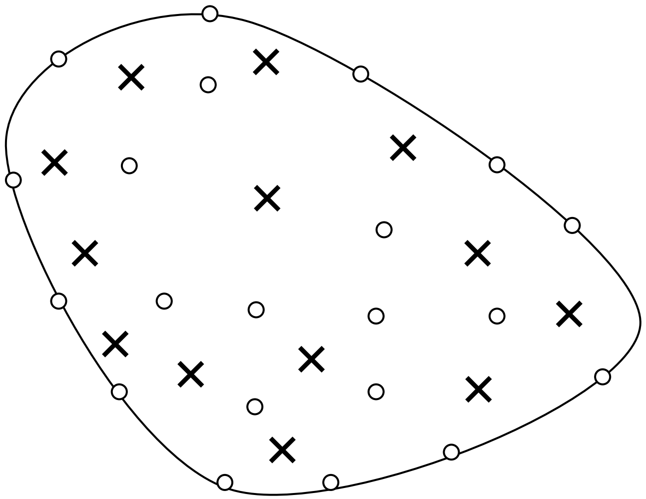
- **smoothing** of the direct EA solution,
- **balancing** of smoothed EA solution,
- step by step **mesh refinement**,
- use of **a' posteriori error analysis** and non-standard **parallel** and **distributed** calculations, combined with **mesh refinement**.

Development of chosen **existing** techniques:

- **new** evolutionary **operators** (e. g. gradient mutation, cloning),
- **hybrid** algorithms (EA + deterministic method).
- **distributed** and **parallel** algorithms.

# Step by step mesh refinement

**Adaptive** approach



○ - nodes of **old** cloud (with known solution)

× - candidates for the **new** cloud

Generation criterion of new nodes

**error** $\boxed{\overline{e}_x > \tau \cdot \overline{e}_{\max} \;,\; 0 < \tau < 1}$

**mesh density gradient** $\boxed{\dfrac{\left|d_i - d_j\right|}{\rho_{ij}} > \theta_{adm}}$

Break off criterion

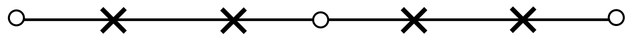
$$\boxed{\forall\ \overline{e}_x < e_{adm}}$$

Use of **a'posteriori error analysis** for estimating the quality of nodes distribution and adaptation approach

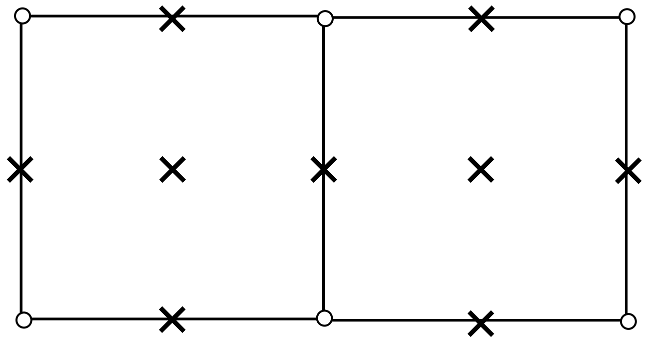
# Step by step mesh refinement

**Regular** meshes (for benchmark problems)

**1D**



**2D**



**Initial** function values in inserted nodes may be calculated using **interpolation** spanned over coarse mesh

○ - nodes of **old** cloud

× - **new** nodes

# Smoothing by 1D MWLS approximation technique

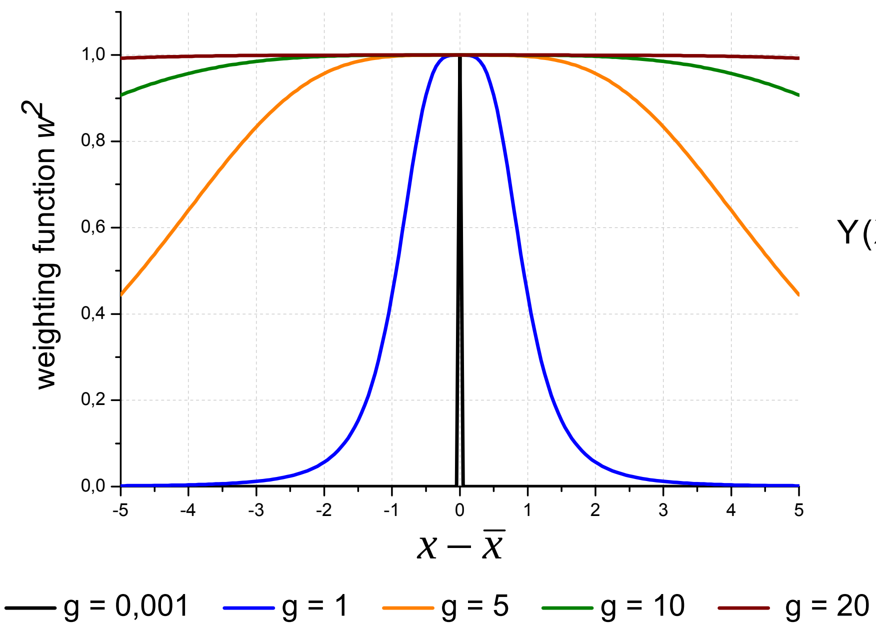
Weighted error **functional** $B = \sum_{i=1}^{n} (u_i - \bar{u}_i)^2 w_i^2$ , where $w_i^2 = \left( h_i^2 + \frac{g^4}{h_i^2 + g^2} \right)^{-p-1}$

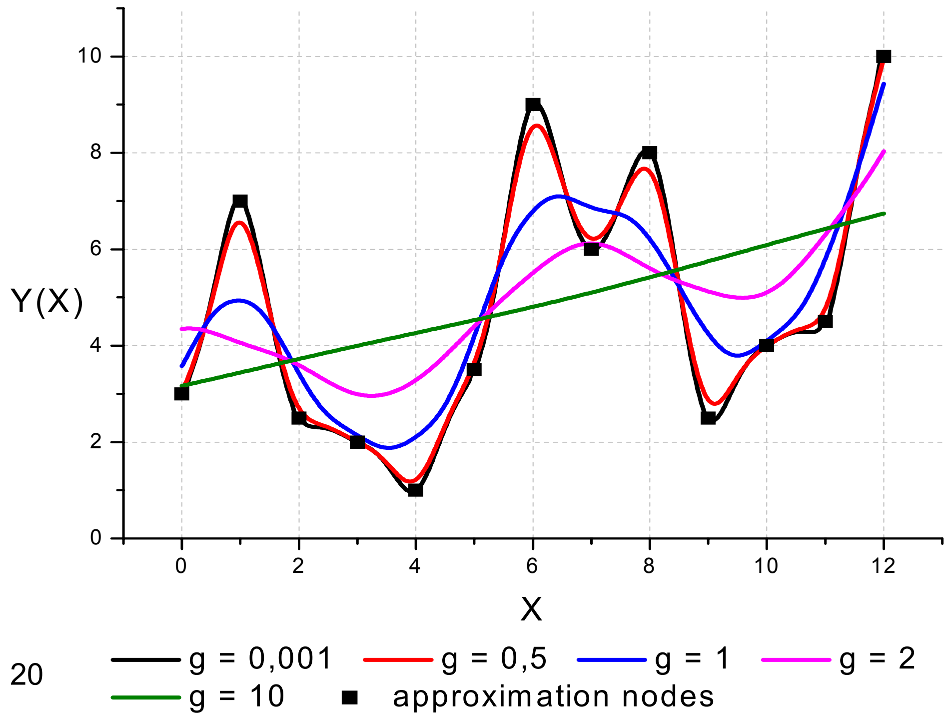
$$\bar{u}_i = \bar{u}(x_i) = \bar{u} + h_i \bar{u}' + \frac{1}{2} h_i^2 \bar{u}'' + \ldots + \frac{1}{p!} h_i^p \bar{u}^{(p)} + R \qquad \bar{u} = \bar{u}(\bar{x}) \qquad h_i = |\bar{x} - x_i|$$

Minimization **conditions**: $\frac{\partial B}{\partial \bar{u}} = 0, \quad \frac{\partial B}{\partial \bar{u}'} = 0, \quad \frac{\partial B}{\partial \bar{u}''} = 0, \quad \ldots, \quad \frac{\partial B}{\partial \bar{u}^{(p)}} = 0$

$$\Rightarrow \boxed{\bar{u},\ \bar{u}',\ \bar{u}'',\ \ldots,\ \bar{u}^{(p)}}$$

Weighting function for various $g$ values:



# A' posteriori solution error analysis

**Solution** of optimization problem

$u^*$ **true**

$\tilde{u} = \tilde{u}_{h,p}$ **rough** solution

$\overline{u}$ improved **reference** solution - **hierarchic** approach

**new solutions and postprocessing required**

**Local** a' posteriori **error**

$e_T = \tilde{u} - u^*$ **true** error

$e_E = \tilde{u} - \overline{u}$ **estimated** error

# A’ posteriori solution error analysis

**Global** a’ posteriori solution error

$e = \|e_T\|$ **true** error

$\eta = \|e_E\|$ **estimated** error

Error **estimation** type: **hierarchic** - h, p, smoothed HO

Error norms

**mean square:** $e_{MEAN} = \left( \frac{1}{\Omega} \int_{\Omega} e_T^2 \, d\Omega \right)^{\frac{1}{2}}$ $\eta_{MEAN} = \left( \frac{1}{\Omega} \int_{\Omega} e_E^2 \, d\Omega \right)^{\frac{1}{2}}$

**maximum:** $e_{MAX} = \max_{x \in \Omega} e_T$ $\eta_{MAX} = \max_{x \in \Omega} e_E$

Estimation quality – **effectivity index** $i = 1 + \frac{\left| \|e\| - \|\eta\| \right|}{\|e\|}$

Fitness function value $f_{best}$ control $\xi = \left| \frac{f_{best}^{(m)} - f_{best}^{(m-1)}}{f_{best}^{(m)}} \right| \leq \xi_{admissible}$

# Generation of reference solutions

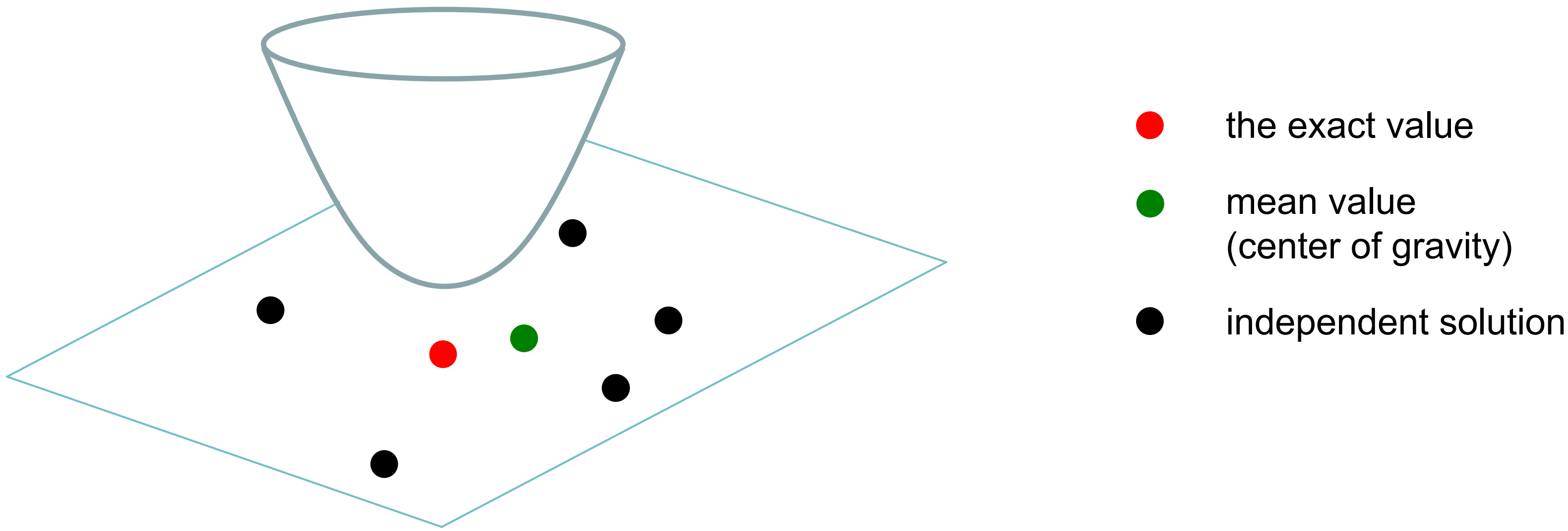
## In the **Evolutionary Algorithms**

### Approach

- generation of population of $m$ **independent** solutions (chromosomes),
- **weighted averaging** of these results over the whole population,
- **postprocessing** (HO smoothing) of the above discrete solution by means of the MWLS (or PBA) approximation,

### Notice

**h** - type and/or **p** - type estimation approach would be also possible.



## Use of a' posteriori error analysis for the EA solution process **acceleration**

Approach **concept**

- Find or estimate EA solution **error** in the **whole** domain.
- **Intensify calculations** in zones where solution **error** is **larger** than elsewhere.

  Do this by means of **modification** of the basic EA operators:

  **selection**, **mutation** , and **crossing-over**.

Modification of the **selection** operator

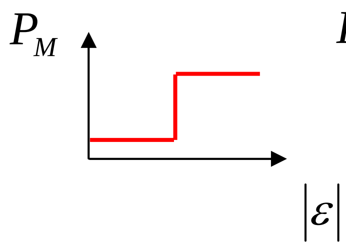
Examine various ways of simultaneous use of two selection criteria

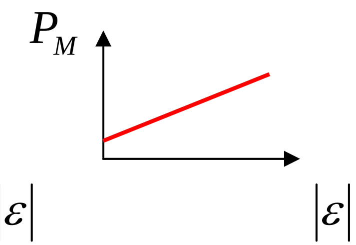
- fitness function value $f$ ,
- solution error value $e$ .

## Modification of the **mutation** operator

Increasing **probability** of mutation:

- in regions where error is **larger** than e.g. ½ mean error,
- **following function** of the error level.




## Modification of the **crossing-over** operator

For heuristic **crossing-over**:

- crossing-over in **direction** of the **best** of **three** chromosomes:

  mean M and two parents X, Y,

  e.g. If F(M) > F(P) :

  $$Z = a(M - P) + P, \qquad P = \sup\{X, Y\}$$

  otherwise

  standard procedure

**Standard** heuristic crossing-over:

X, Y - parents,
Z - offspring,
F - fitness function.

If F(X) > F(Y)

$$Z = a(X - Y) + X$$

$$a \in [0,1]$$

- increasing **value** of the parameter $\boxed{a}$.

Other: **increasing** crossing-over **probability** for values larger than e.g. ½ mean error.

# Use of mesh refinement with a’posteriori error estimation

## Strategy

(i) calculation of solution on a coarse mesh

(ii) smoothing of rough solution

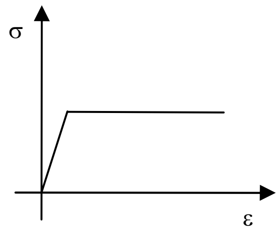
(iii) mesh refinement and the best approximation (or interpolation) of initial function values at inserted nodes

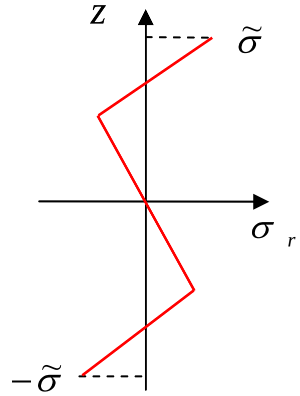
(iv) use of obtained solution as the initial reference solution for a’posteriori error estimation

(v) repetition of the procedure given above until a sufficiently dense mesh is reached

# Benchmark tests

## (1) Residual stress analysis in a bar subject to pure cyclic bending – exact 1D model




Problem formulation:

Find $\min\limits_{\sigma} I$

Fitness function: $\min\limits_{\sigma} \int_0^H \sigma^2 dz$ complementary energy

constraints: $M = \int_0^H \sigma \, z dz = 0$ global equilibrium

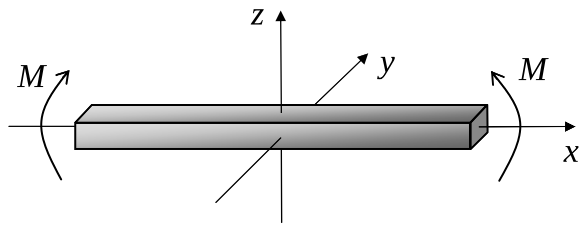
$\left| \sigma + \sigma^e \right| \le \sigma_y$ yield condition for total stresses

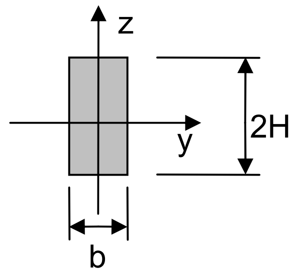
where:

$\sigma_y$ – yield stress

$\sigma^e$ – elastic solution of the problem

E – Young modulus



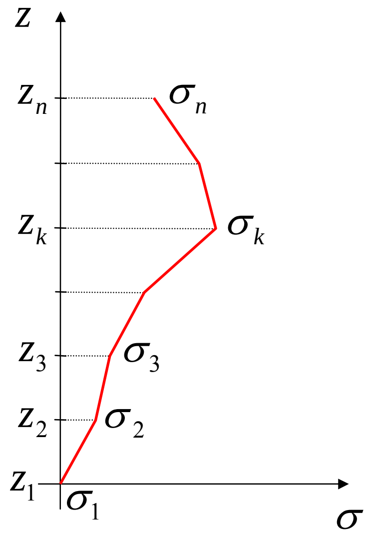




NOTICE :

- **EXACT** SOLUTION IS KNOWN
- **ANY NUMBER** OF DECISION VARIABLES MAY BE CHOSEN (IN 1D OR 2D)

Final discrete **1D** formulation of the problem:

Find stresses $\sigma_1, \sigma_2, \ldots, \sigma_n$ satisfying:




$$\min_{\sigma_1, \ldots, \sigma_{n-1}} \left( \sum_{k=1}^{n-1} \sigma_k^2 + \frac{1}{2} \sigma_n^2 \right)$$

minimum of the total complementary energy

$$\sigma_n = -\frac{2}{z_n} \sum_{k=1}^{n-1} \sigma_k z_k$$

global equilibrium equation

$$-1 \le \frac{\sigma_k}{\sigma_Y} - \frac{3}{2}\left[1 - \frac{1}{3}\left(\frac{\bar{Z}}{H}\right)^2\right]\frac{k-1}{n-1} \le 1$$

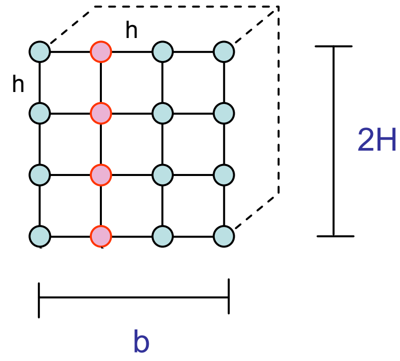
condition for total stresses (plastic limit)

$$k = 1, \ldots, n$$

Data assumed: $\sigma_Y = 1$ ; $\dfrac{\bar{Z}}{H} = \dfrac{1}{2}$ .

# Benchmark tests

## (2) Discrete 2D model




**Fitness** function:

Total complementary energy

$$I = \int_{-b/2}^{b/2} \int_{-H}^{H} \sigma^2 \, dz dy \approx \frac{h^2}{9}\left( \sum_{k=1}^{n} \sigma_k^2 \alpha_k \right), \qquad \alpha_j \; - \text{ integration coefficient}$$

**Constraints:**

bending moment – equilibrium equation

$$M = \int_{-b/2}^{b/2} \int_{-H}^{H} \sigma \, z \, dz dy \approx \frac{h^2}{9}\left( \sum_{k=1}^{n-1} \sigma_k z_k \alpha_k + \sigma_n z_n \alpha_n \right) = 0 \;\rightarrow\; \sigma_n = \frac{\sum_{k=1}^{n-1} \sigma_k z_k \alpha_k}{z_n \alpha_n}$$

yield condition constraints

$$-1 \le \frac{\sigma_k + \sigma_k^e}{\sigma_Y} \le 1, \qquad k = 1, \ldots, n$$

$$\sigma_k^e = \frac{3}{2} \frac{\sigma_Y}{H}\left[ 1 - \frac{1}{3}\left( \frac{\bar{Z}}{H} \right)^2 \right] z_k \qquad z_k = k \frac{H}{n}$$

$b \times 2H$ – size of beam cross-section

$\sigma^Y$ – yield stress

$\sigma^e$ – purely elastic solution stress

# Benchmark tests

## (3-4) Residual stresses analysis in thick-walled cylinder under cyclic pressure

### Exact (axially symmetrical) formulation

**Find** minimum of the total **complementary energy**

$$\min_{\sigma_r^r, \sigma_t^r, \sigma_z^r} \Pi , \qquad \Pi = \tfrac{1}{2}\int_a^b [(\sigma_r^r - \sigma_t^r)^2 + (\sigma_t^r - \sigma_z^r)^2 + (\sigma_z^r - \sigma_r^r)^2 ]\, r\, dr$$

**subject to**

**equilibrium** equation

$$\frac{\partial \sigma_r^r}{\partial r} + \frac{\sigma_r^r - \sigma_t^r}{r} = 0$$

**boundary** conditions

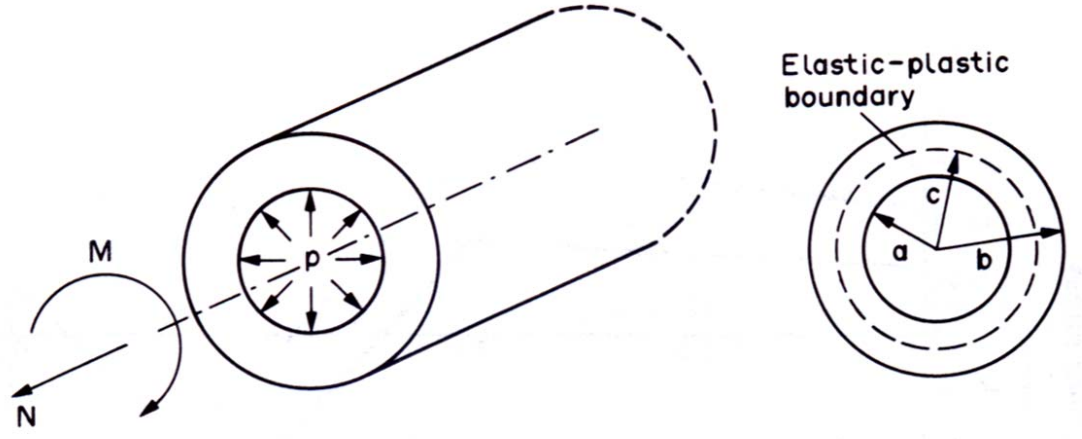
$$\sigma_{r|a}^r = 0, \quad \sigma_{r|b}^r = 0$$

**incompressibility** equation

$$\sigma_z^r = \nu(\sigma_r^r + \sigma_t^r)$$

**yield** condition

$$\phi(\sigma_r^r, \sigma_t^r, \sigma_z^r, \sigma^E) \le \sigma_Y$$

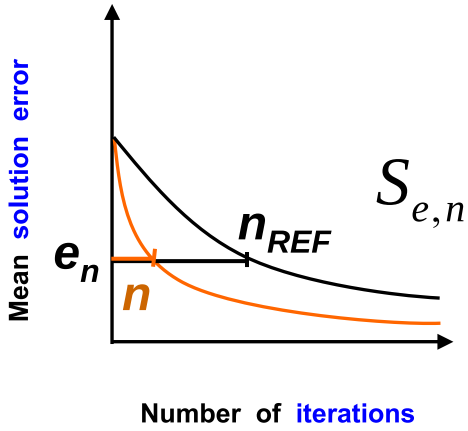



**Benchmark (3) - model 1D**

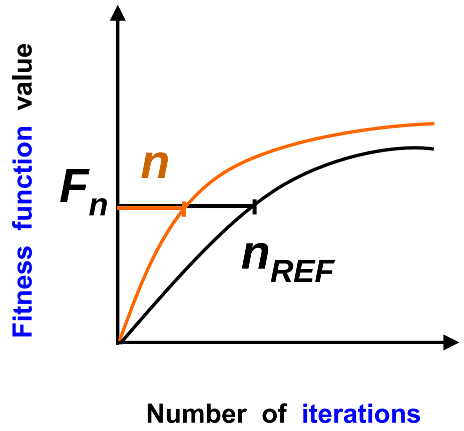
**Benchmark (4) - model 2D**

# Preliminary results for benchmark problems – introduction

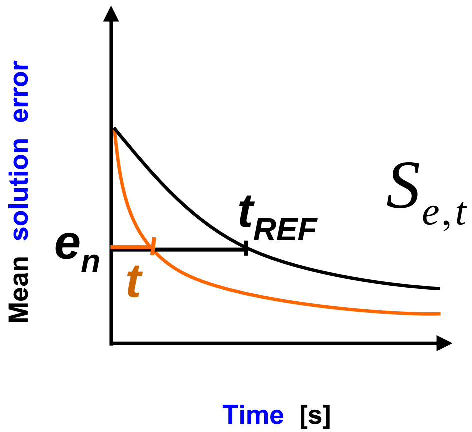
Four types of charts for **speed-up** $S$ determination



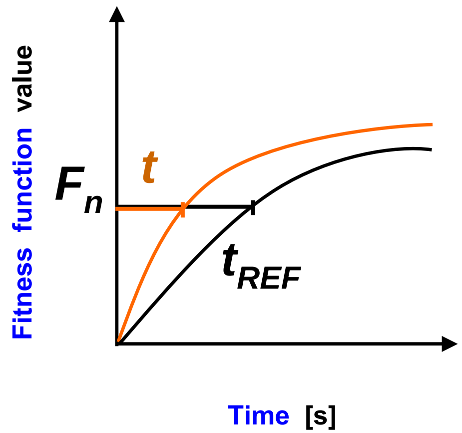

$$S_{e,n} = \left.\frac{n_{REF}}{n}\right|_{e=const}$$




$$S_{F,n} = \left.\frac{n_{REF}}{n}\right|_{F=const}$$




$$S_{e,t} = \left.\frac{t_{REF}}{t}\right|_{e=const}$$

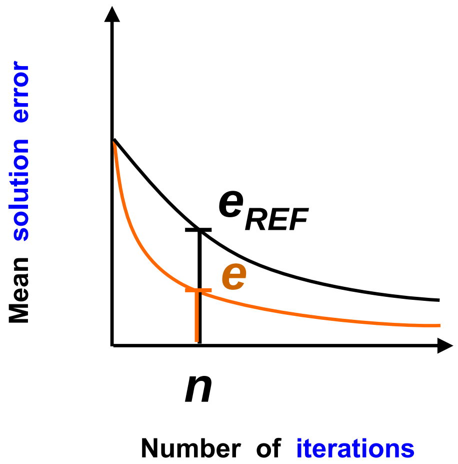



$$S_{F,t} = \left.\frac{t_{REF}}{t}\right|_{F=const}$$

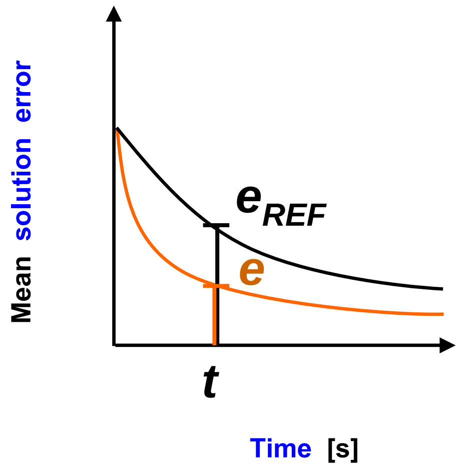
All four above situations together provide the **full** acceleration **characteristics** investigated.

# Preliminary results for benchmark problems – introduction

Two types of charts for **precision increase** $P$ determination

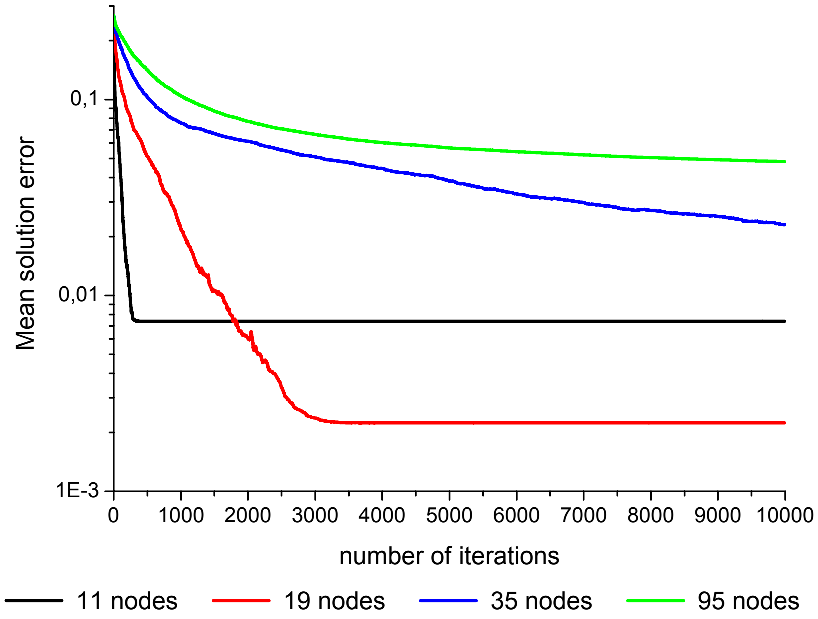


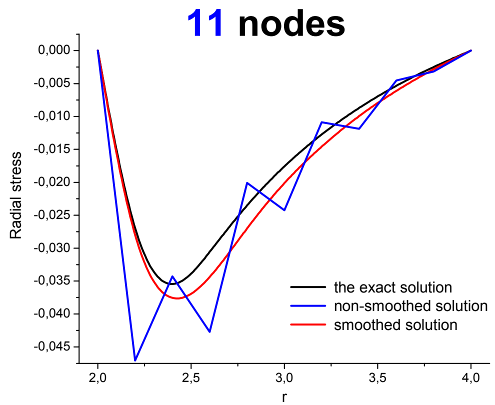

$$P_n = \frac{e_{REF}(n)}{e(n)}$$

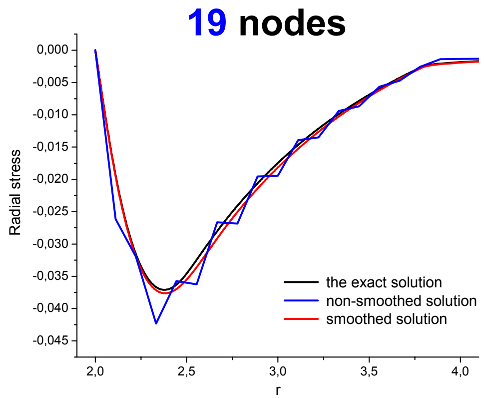


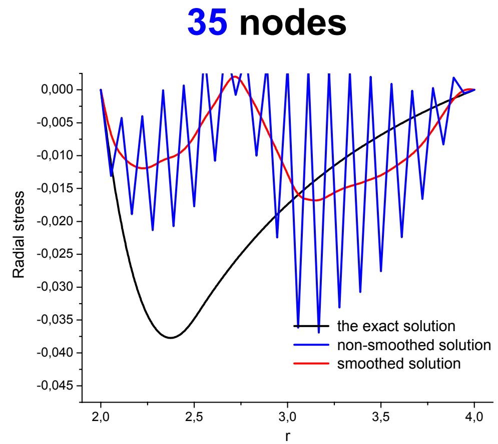

$$P_t = \frac{e_{REF}(t)}{e(t)}$$

# T1. Motivation example



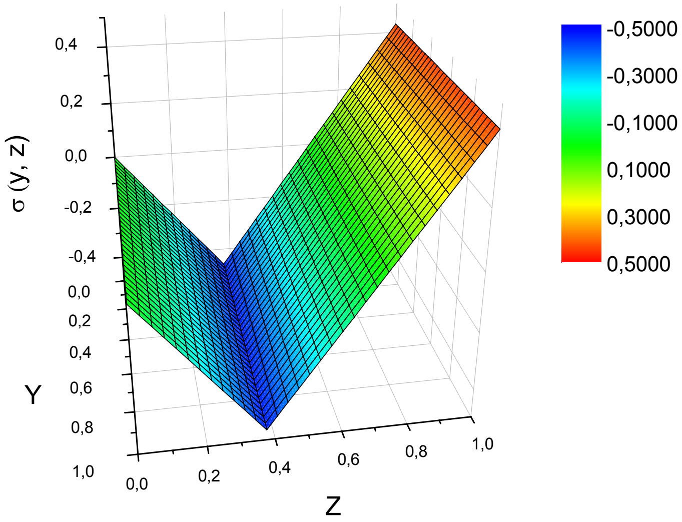



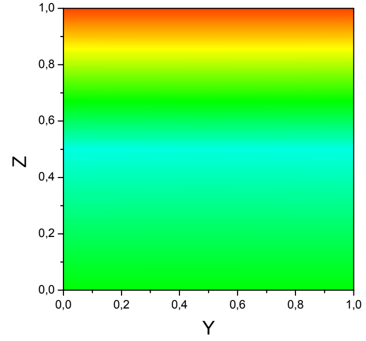




**Impact of mesh density on the convergence rate and solution precision**

**Benchmark 3**

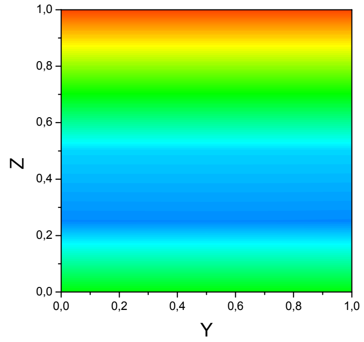
# T2. Examination of the possibility of solving large optimization problems using mesh refinement and error estimation: benchmark (2)



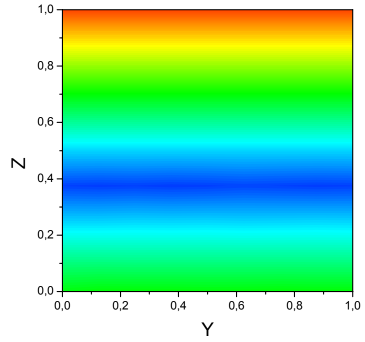

**9 nodes**

**3 x 3**



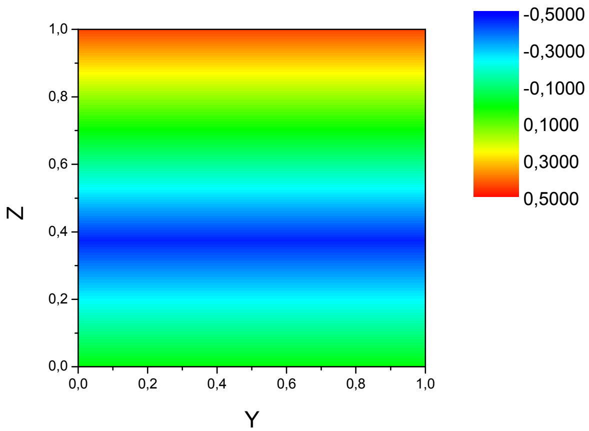

**25 nodes**

**5 x 5**

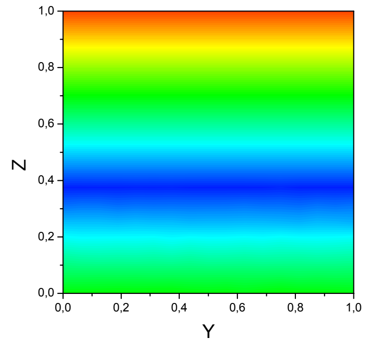



**81 nodes**

**9 x 9**

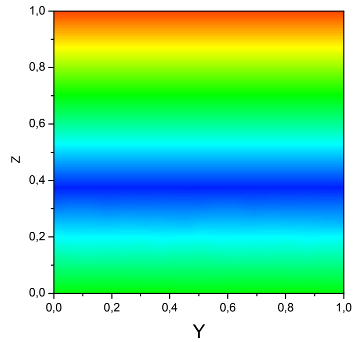



**the true solution**




**289 nodes**

**17 x 17**

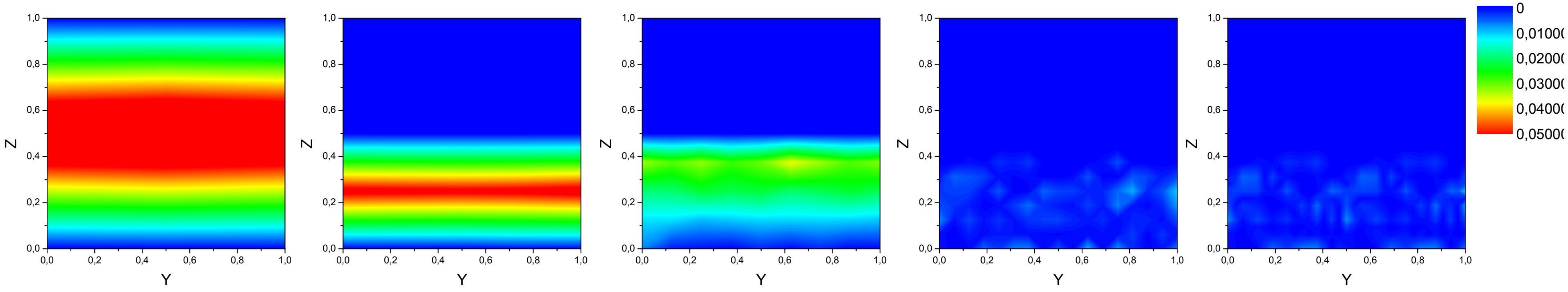



**1089 nodes**

**33 x 33**

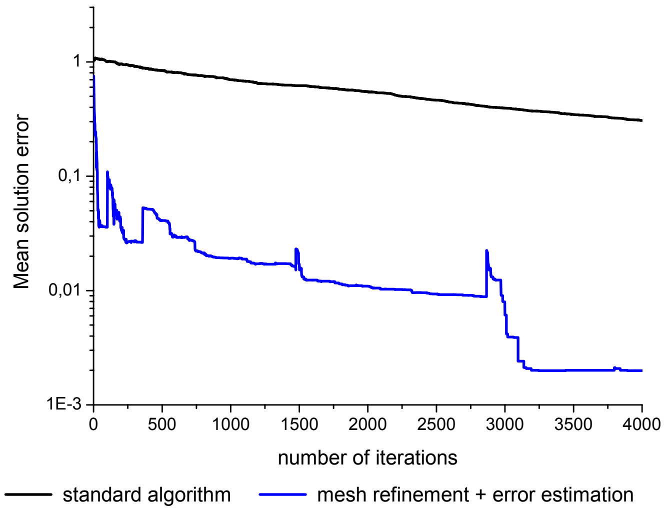
# T2. Examination of the possibility of solving large optimization problems using mesh refinement and error estimation: benchmark (2)

## The true error map




9 nodes | 25 nodes | 81 nodes | 289 nodes | 1089 nodes

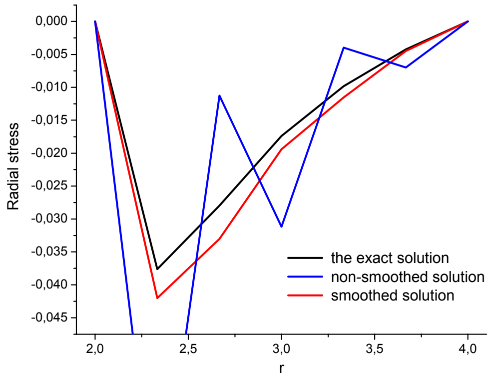
## Convergence of mean solution error – comparison with standard algorithm



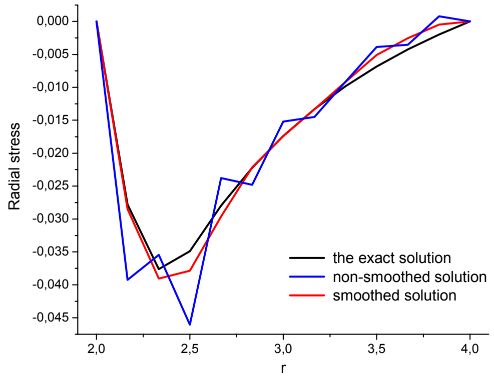

**Speed-up: about 20 - 400 times**
**Precision increase: about 150 times**

# T3. Examination of the possibility of solving large optimization problems using mesh refinement and error estimation: benchmark (3)

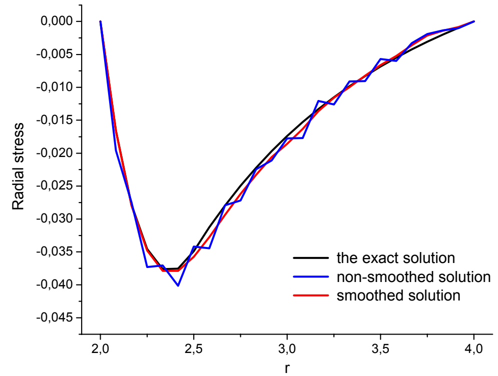
## 1D model



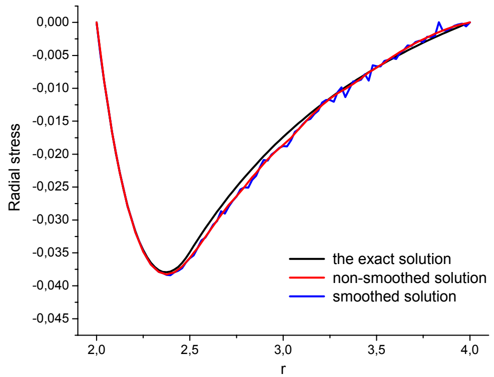

**7 nodes**



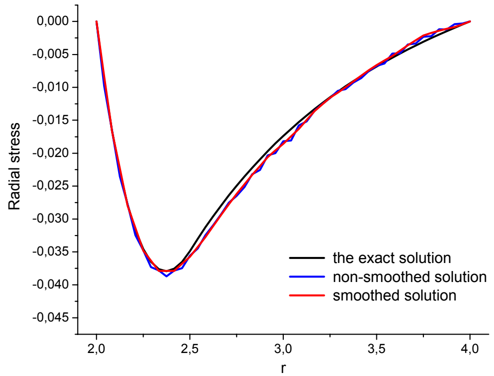

**13 nodes**




**25 nodes**




**49 nodes**




**97 nodes**

### Errors

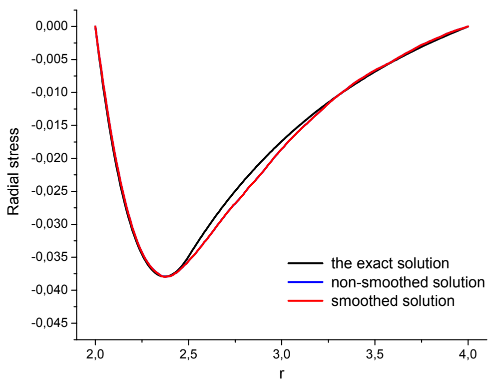
| mesh | mean err. | max err. |
|---|---|---|
| 7 | 0.0208 | 0.0408 |
| 13 | 0.0053 | 0.0114 |
| 25 | 0.0016 | 0.0033 |
| 49 | 0.0011 | 0.0023 |
| 97 | 0.0008 | 0.0018 |

# T3. Examination of the possibility of solving large optimization problems using mesh refinement and error estimation: benchmark (3)

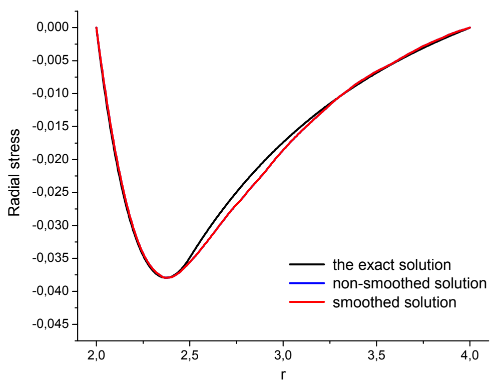
**1D model**

**Continuation of mesh refinement:**

**97 → 193 → 385 → 769 → 1537 → 3073 nodes**



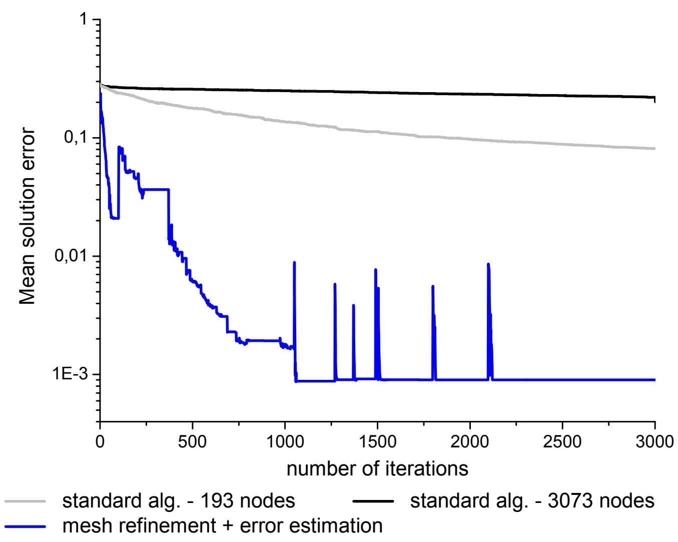

**193 nodes**




**3073 nodes**

| | | |
|---|---|---|
| **Nodes** | 193 | 3073 |
| **Mean err.** | 0.0009055 | 0.000903 |
| **Max err.** | 0.001984 | 0.001980 |

**convergence of mean solution error – comparison with standard algorithm**



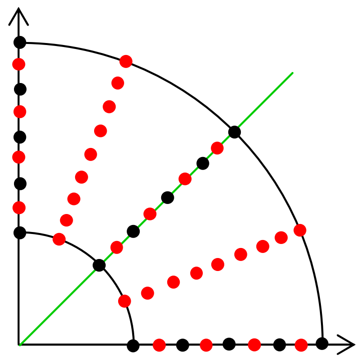

**Speed-up: about 120 times**

**Precision increase: about 90 times**

# T4. Examination of the possibility of solving large optimization problems using mesh refinement and error estimation: benchmark (4)

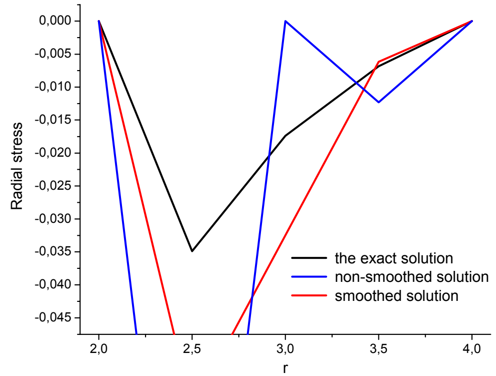
**2D model**

## Discretization strategy



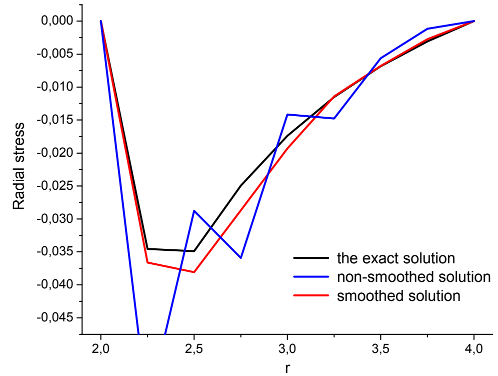
- nodes of **old** mesh
- **new** nodes

## Solution in cross-section



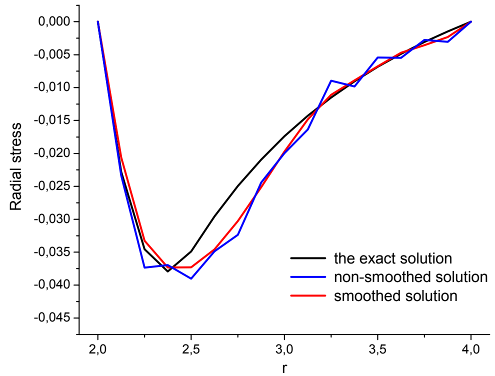

**15 nodes**

**5 x 3**



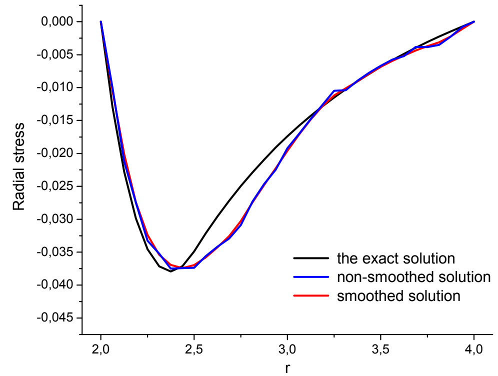

**45 nodes**

**9 x 5**




**153 nodes**

**17 x 9**



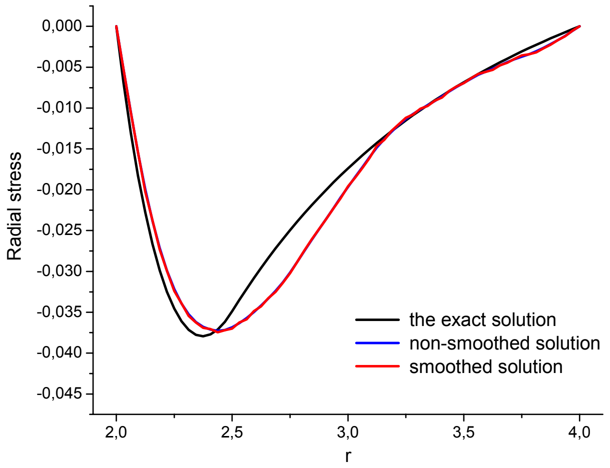

**561 nodes**

**33 x 17**

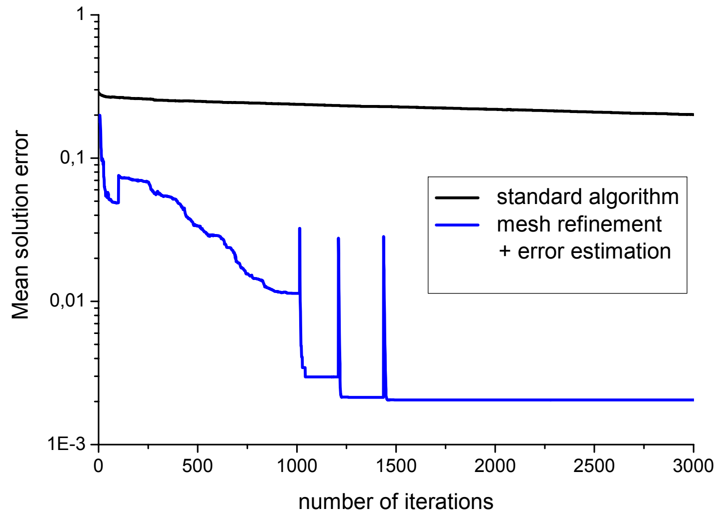
# T4. Examination of the possibility of solving large optimization problems using mesh refinement and error estimation: benchmark (4)

**2D model**

**solution in cross-section for 2145 nodes and mesh 65 x 33**

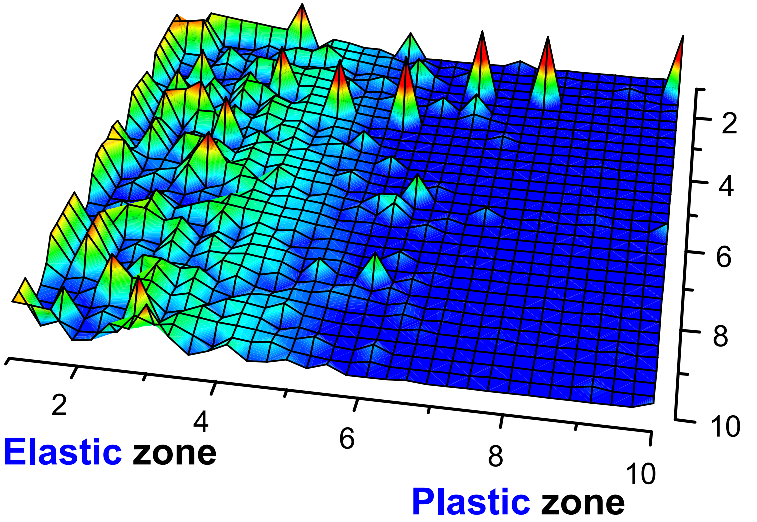



**convergence of mean solution error – comparison with standard algorithm**

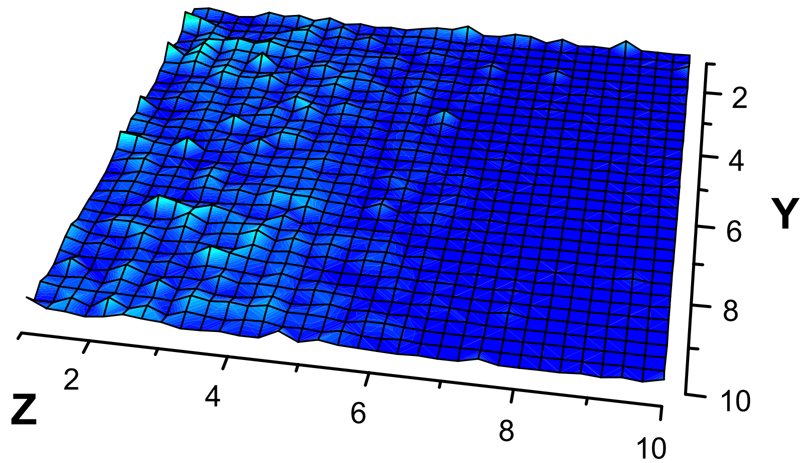


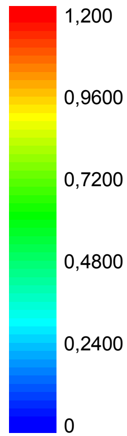

**Speed-up: about 150 - 300 times**
**Precision increase: about 100 times**

# T5. Comparison of the exact and estimated stress errors for benchmark (2) after 3000 iterations

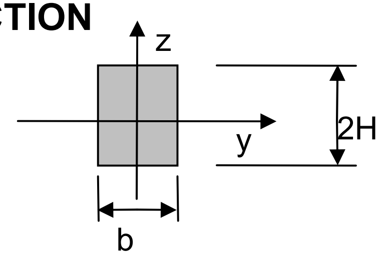


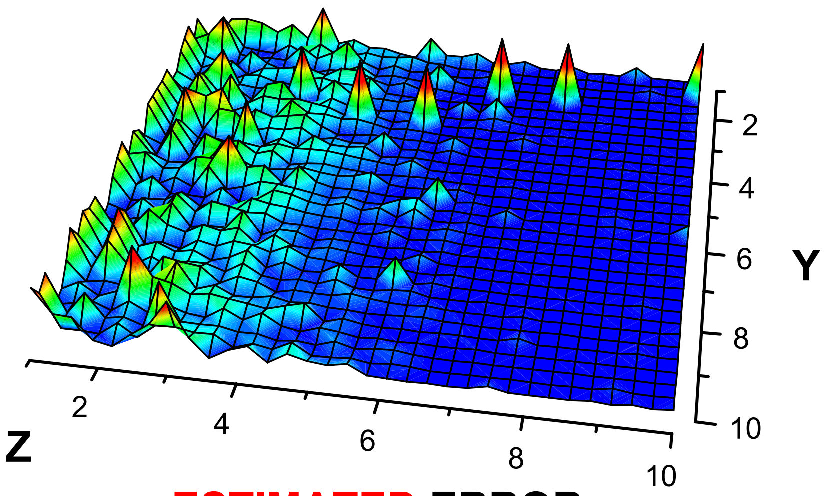

**EXACT ERROR**




**DIFFERENCE BETWEEN THE EXACT AND ESTIMATED ERROR IN BEAM CROSS-SECTION**




**ESTIMATED ERROR**

reference solution obtained by averaging 12 independent chromosomes

**Mean values:**

| | $e_i$ | $e_i / e_{exact}$ |
|---|---|---|
| **exact** | **0.338** | **1** |
| **estimated** | **0.326** | **0.96** |
| **difference** | **0.095** | **0.28** |

**effectivity index** $$i = 1 + \frac{\left| \|e\| - \|\eta\| \right|}{\|e\|} = 1.04$$

# Final remarks

## Summary

- Many **scientific** and **technical** tasks may be expressed in terms of **non-linear**, **constraint optimization problems**. In a wide class of such problems the objective is to find an **unknown function**, mostly in a discrete form.

- **Optimization** problems may be solved by means of either **deterministic** or **probabilistic** methods. The first ones are very effective when dealing with the **convex** problems as opposed to usually slowly convergent **probabilistic** e.g. most AI methods, especially for **large** optimization problems. However, their **efficiency** does **not change** much for **non-convex** methods as opposed to the deterministic ones.

- The **objective** of this whole research, therefore, is to develop means of **essential acceleration** of the **Evolutionary Algorithms**, one of the AI methods. Particular attention is paid here to use step by step **mesh refinement** and use our knowledge about estimated **solution error** to the **EA** acceleration. These concepts are tested on various, carefully selected **benchmark** problems, using the **true** solution error, replaced later on by the **estimated** errors. Solutions based on **series of more and more dense meshes** are used as the reference ones then.

## Summary

- Very preliminary **results** of these tests are encouraging. For use of step by step **mesh refinement** together with a'posteriori **error analysis**, the overall speed-up factor about **150** times was reached.

- When well designed, the **EA speed-up** may provide solution much **faster** than the **standard EA** algorithm. Moreover, as opposed to the standard approach such solution may also be obtained for **large** optimization problems.

## **Further research** planned

- continuation of various efforts oriented towards **increasing** the **EA efficiency**, including testing further concepts, and combining all types of acceleration together for their simultaneous use;

- analysis of further **benchmarks**;

- application to **residual stress** analysis in **railroad rails** and **vehicle wheels**;

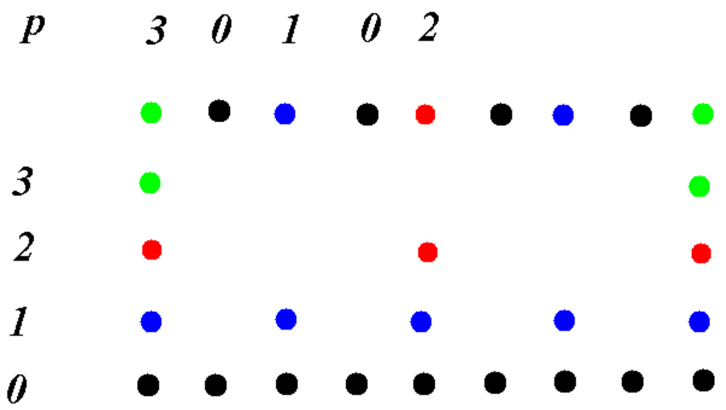
- analysis of **large**, **non-linear**, **constrained optimization problems** (convex and non-convex) resulting from the physically based **approximation** applied to **experimental** measurements.

**THANK YOU VERY MUCH**
**FOR ATTENTION**

# Generacja węzłów

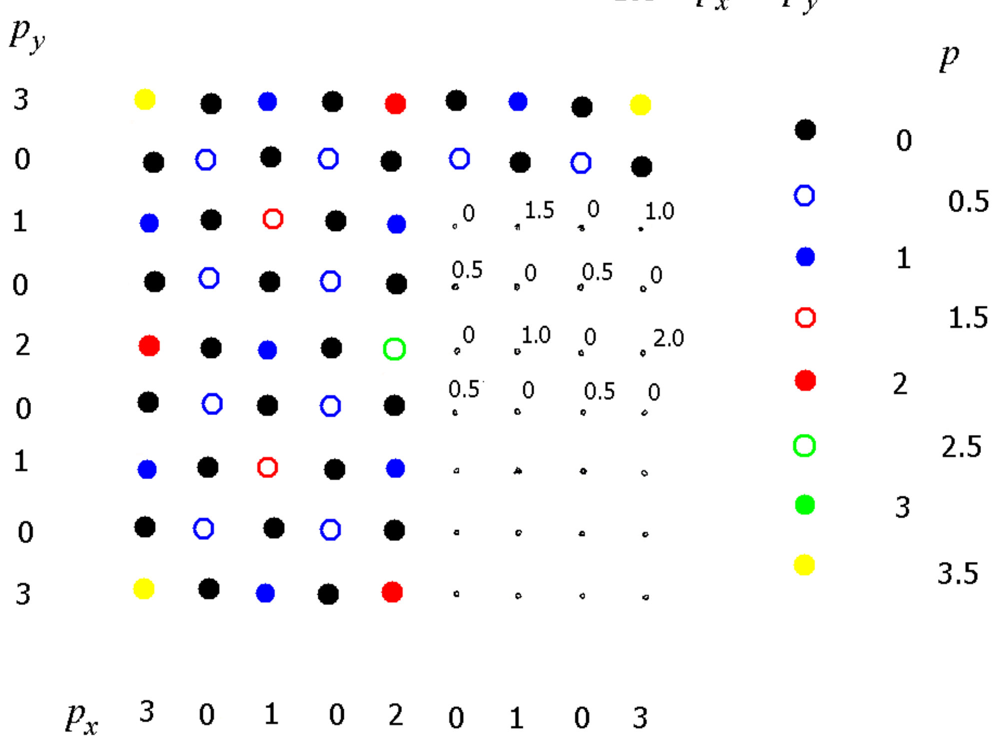
## siatka 1D

$$p = \log \frac{r}{r_{\min}}$$




## siatka 2D

$$p = \begin{cases} min\,(p_x, p_y) & \text{for} \quad p_x \neq p_y \\ p_x + 0.5 & \text{for} \quad p_x = p_y \end{cases}$$



# Generacja siatki metodą Liszki

- Zdefiniowanie lokalnej gęstości siatki regularnej

$$\rho^{-1} \equiv \log_2 \frac{r}{r_{ref}} = \log_2 \frac{r}{r_{\min}} + \log_2 \frac{r_{\min}}{r_{ref}}$$

$$p = \log_2 \frac{r}{r_{\min}}; \qquad \rho_{\max}{}^{-1} = \log_2 \frac{r_{\min}}{r_{ref}}$$

and $r_{ref} \leq r_{\min}$

jest założonym najmniejszym modułem siatki regularnej

stąd

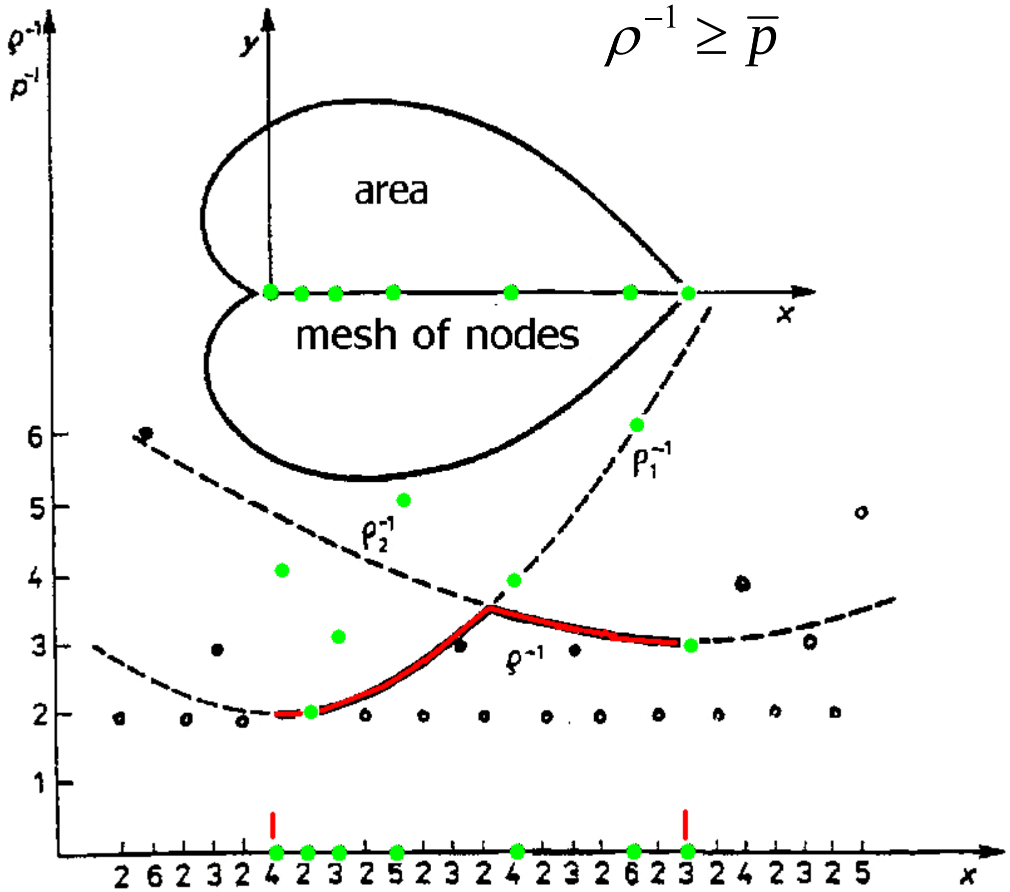
$$\rho^{-1} - \rho_{\max}{}^{-1} = p$$

- Przypisanie żądanej gęstości $\overline{p}$
- Wybór węzłów z siatki na podstawie kryterium

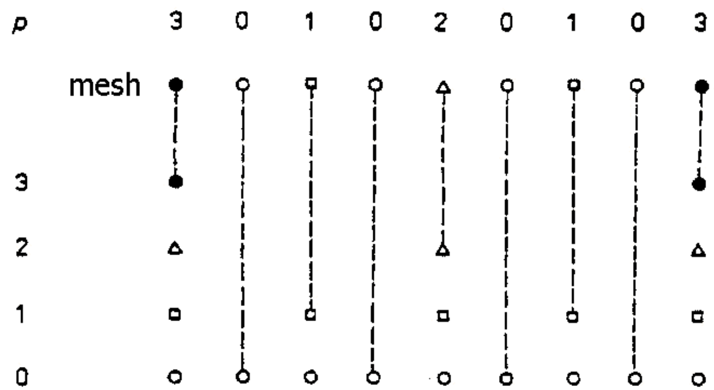
$$\boxed{\rho^{-1} - \rho_{\max}^{-1} \geq \overline{p}}$$

# Przykład 1D

$$\rho^{-1} \geq \overline{p}$$

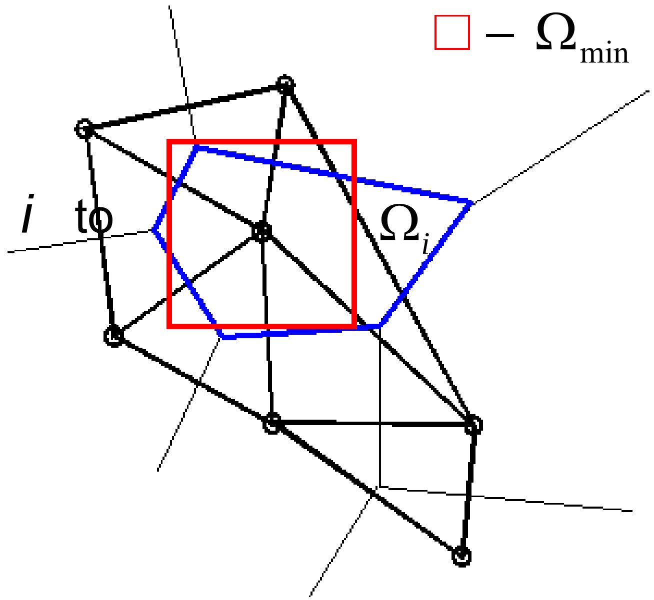



numbers characterizing local reversed mesh density factors




weighting factor p for 1D mesh

# Definicja gęstości siatki nieregularnej




- Gęstość siatki
  - gęstość siatki $\rho_i$ w dowolnym węźle $i$ to

$$\rho_i^{-1} = \rho_{\max}^{-1} + p_i \ ,$$

gdzie

$$p_i = \log_2 \left( \frac{\Omega_i}{\Omega_{\min}} k \right)^{1/2}$$

  - gęstość siatki $\rho$ w dowolnym punkcie $\boldsymbol{x}$ obszaru jest obliczania

przez aproksymację MWLS

$$\rho = \sum_{star} \rho_i \Phi_i ,$$

gdzie $\Phi_i = \Phi_i(x)$ są pseudo – funkcjami kształtu MLWS